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APPROXIMATING ALGEBRAIC RICCATI EQUATIONS IN INFINITE DIMENSIONS

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1. INTRODUCTION, NOTATION AND HYPOTHESES

This work concerns the convergence of solutions to algebraic Riccati equations:

(1.1) $$A_h^* P_h + P_h A_h - P_h B_h B_h^* P_h + C_h^* C_h = 0,$$

to the solution P to the equation:

(1.2) $$A^* P + PA - PBB^* P + C^* C = 0,$$

when $\{A_h\}$, $\{B_h\}$, $\{C_h\}$ are converging in the sense of graph to A, B, and C respectively, when $h \searrow 0$. Equations (1.1), (1.2) are relevant in feedback stabilization of the linear dynamic system:

(1.3) $$x'(t) = Ax(t) + Bu(t), \quad x(0) = x_0.$$

The main results assert that under suitable assumptions, the solutions to (1.1) converge in a weak sense to the solution to (1.2), and $u_h = -B_h^* P_h x$, for h small, is a stabilizing feedback controller for system (1.3).

An application to a control system governed by a functional differential equation is given.

Consider three Hilbert spaces together with their norms: $(X, \|\cdot\|)$ – the state space, $(U, |\cdot|)$ – the control space, and $(Z, \|\cdot\|_Z)$ – the observation space.

As in [11], $G(M,\omega)$ denotes the class of operators A: $D(A) \subseteq X \to X$ which are infinitesimal generators of C_0–semigroups, $\{S(t);\, t \geq 0\}$, satisfying $\|S(t)\| \leq M \exp(\omega t)$, where M and ω are real constants, $M \geq 1$, $\omega \geq 0$.

Consider the following regulator problem: given the dynamical system (1.3), minimize the quadratic functional:

$$J(u,x)=\int_0^{\infty}\left(\|Cx(t)\|_Z^2+|u(t)|^2\right)\mathrm{d}t, \tag{1.4}$$

over all $u \in L^2(0, +\infty; U)$ and x solution to (1.3) corresponding to u.

Related to this control problem, we consider the following approximating quadratic problem: minimize the functional:

$$J_h(u,x_h)=\int_0^{\infty}\left(\|C_h x_h(t)\|_Z^2+|u(t)|^2\right)\mathrm{d}t, \tag{1.5}$$

over all $u \in L^2(0, +\infty; U)$ and x_h solution corresponding to u of the approximate dynamical system:

$$x'_h(t)=A_h x_h(t)+B_h u(t),\quad x_h(0)=x_0, \tag{1.6}$$

where h is a small parameter.

For the operators considered above, we make the following assumptions:

(i) $A, A_h \in G(M,\omega)$, for all h, and there exists $\lambda_0 \in \mathbf{C}$, $\Re\lambda_0 > \omega$ such that:

$$R(\lambda_0; A_h)x \to R(\lambda_0; A)x,\ \text{when } h \searrow 0, \text{ for all } x\in X. \tag{1.7}$$

($R(\lambda; A)$ is the resolvent operator associated to A, i.e., $(\lambda I - A)^{-1}$.

(ii) B, B_h, C, C_h are linear bounded operators for all h, $B, B_h \in \mathcal{L}(U,X)$, $C, C_h \in \mathcal{L}(X,Z)$, and:

$$B_h u \to Bu,\ B^*_h x \to B^* x \text{ as } h \searrow 0, \tag{1.8}$$

for all $u \in U$ and $x \in X$,

$$C^*_h C_h x \to C^* C x, \text{ as } h \searrow 0, \text{ for all } x \in X. \tag{1.9}$$

(iii) (detectability)

There exist $K, K_h \in \mathcal{L}(Z,X)$, linear bounded operators such that the operators $A + KC$ and $A_h + K_h C_h$ generate exponentially stable semigroups, for all h.

(iv) (uniform stabilizability)

There exists $F \in \mathcal{L}(X,U)$ a bounded linear operator, s.t. $A_h + B_h F$, and $A + BF$ generate exponentially stable semigroups, $\{S_{h,F}(t);\ t \geq 0\}$, and $\{S_F(t);\ t \geq 0\}$ respectively, when h is small enough, i.e., there exist two real constants, $M_1 \geq 1$, $\omega_1 \geq 0$ such that:

$$\|S_{h,F}(t)\| \leq M_1 \exp(-\omega_1 t),\quad \text{for all } t>0, \tag{1.10}$$

$$\|S_F(t)\| \leq M_1 \exp(-\omega_1 t),\quad \text{for all } t>0. \tag{1.11}$$

Under these assumptions, it is known ([2]) that the control problems (1.3), (1.4), and (1.5), (1.6) have unique optimal pairs (x^*,u^*), and (x_h^*, u_h^*) respectively, related by the feedback laws:

$$u^*(t) = -B^*Px^*(t),\quad u^*_h(t) = -B^*_h P_h x^*_h(t),\quad t>0, \text{ for all } h, \tag{1.12}$$

where $P, P_h \in \mathcal{L}(X)$ are linear bounded selfadjoint, positive operators, solutions to the algebraic Riccati equations (1.2) and (1.1) respectively. We also have:

$$(Px_0, x_0) = \frac{1}{2}J(u^*, x^*),\quad \text{for all } x_0 \in X, \tag{1.13}$$

$$(P_h x_0, x_0) = \frac{1}{2}J_h(u_h^*, x_h^*),\quad \text{for all } x_0 \in X, \tag{1.14}$$

where J and J_h are given in (1.4) and (1.5) respectively, $x_0 = x^*(0) = x^*_h(0)$, and $(\cdot,\cdot)$ denotes the inner product in X.

The detectability assumption (iii) ensures that the operators $A - BB^*P$, and $A_h - B_h B^*_h P_h$ generate exponentially stable semigroups.

If $A \in G(M,\omega)$ and $\{\widetilde{S}(t); t \geq 0\}$ is the semigroup generated by $\widetilde{A}$, we say that the operator $\widetilde{A}$, satisfies the *spectral determining growth condition* if (see [13]):

$$\omega_0(\widetilde{S}) = s(\widetilde{A}), \tag{1.15}$$

where

$$\omega_0(\widetilde{S}) = \inf\left\{\frac{\ln\|\widetilde{S}(t)\|}{t};\ t>0\right\}$$

$$s(\widetilde{A}) = \begin{cases} \sup\{\Re\lambda;\ \lambda \in \sigma(\widetilde{A})\} & \text{if } \sigma(\widetilde{A}) \neq \varnothing \\ -\infty & \text{otherwise} \end{cases}$$

and $\sigma(\widetilde{A})$ denotes the spectrum of the operator $\widetilde{A}$. Whenever $\widetilde{A}$ is the infinitesimal generator of a C_0-semigroup, then $s(\widetilde{A}) \leq \omega_0(\widetilde{S})$. The equality holds, for example, when (see e.g. [3]):

1) $\widetilde{A}$ is bounded;
2) There exists $t_0 > 0$ s.t. $\widetilde{S}(t_0)$ is compact;
3) $\{\widetilde{S}(t); t \geq 0\}$ is a differential semigroup;
4) $\{\widetilde{S}(t); t \geq 0\}$ is an analytic semigroup.

2. MAIN RESULTS

The main convergence result is the following:

THEOREM 1. *Assume* (i), (ii), (iii) *and* (iv) *hold. Then:*

$$P_h x_0 \to Px_0,\ \textit{weakly in } X \textit{ as } h \searrow 0, \textit{ for all } x_0 \in X, \tag{2.1}$$

where P and P_h are the linear, bounded, selfadjoint, positive operators, solutions to the Riccati equations (1.2) *and* (1.1) *respectively.*

Proving this result is not enough. In fact, there is a complete theory of convergence for problems of this type (for more details and references see [4]). More interesting, for practical purpose, is to show that the approximate feedback law stabilizes the initial system (1.3), i.e., to prove that the operator $A - BB^*{}_h P_h$ generates an exponentially stable semigroup, for h small enough. When A generates an analytic semigroup, under some natural approximating assumptions, this kind of result was established in [9].

We shall prove a uniform stability result, using the spectral determining growth (s.d.g.) condition. Relation (1.15) tells that one can study the asymptotic behaviour of the semigroup generated by $\widetilde{A}$ only by the knowledge of the spectrum of $\widetilde{A}$, as in finite dimension.

We denote by $\rho(A)$ the resolvent set of the operator A. The uniform stability result derives from the next theorem.

THEOREM 2. *Let $A_1 : D(A_1) \subseteq X \to X$ be an infinitesimal generator of a C_0-semigroup and $\{T_h\} \subseteq \mathcal{L}(X)$ a sequence of linear bounded operators on X. Assume that:*

1. *$A_1 + T_h$ satisfy the s.d.g. condition for all h;*
2. *A_1 generates an exponentially stable semigroup;*
3. *There exists $\lambda_0 \in \rho(A_1)$ s.t. the resolvent operator of A_1, $R(\lambda_0; A_1)$ is compact;*
4. *$T_h x \to 0$ weakly in X as $h \searrow 0$, for all $x \in X$*

Then $A_1 + T_h$ generate exponentially stable semigroups if h is small enough.

COROLLARY. *Assume* (i), (ii), (iii), *and* (iv). *Suppose also that:*

(α) *There exists $\lambda_0 \in \rho(A)$ such that the resolvent operator $R(\lambda_0; A)$ is compact;*

(β) *The operators $A - BB^*{}_h P_h$ satisfy the s.d.g. condition for all h.*

Then $A - BB^*{}_h P_h$ generate exponentially stable semigroups if h is small enough.

Taking $A_1 = A - BB^*P$ and $T_h = BB^*P - BB^*{}_h P_h$ in Theorem 2 we obtain the Corollary.

To prove Theorem 2 we need the following lemma, which probably is not new, but we did not find any mention about it in literature.

LEMMA. *Let $A_1 : D(A_1) \subseteq X \to X$ be a linear closed operator and $\{T_h\} \subseteq \mathcal{L}(X)$ a sequence of linear bounded operators on X. Assume that:*

(a) There exists $\lambda_0 \in \rho(A_1)$ such that the resolvent operator $R(\lambda_0; A_1)$ is compact;

(b) The sequence $\{T_h\}$ is pointwise weakly convergent to 0, *i.e.,*

$$T_h x \to 0, \text{ weakly in } X \text{ as } h \text{ A } 0, \text{ for all } x \in X$$

If $s(A_1) < 0$ then $s(A_1 + T_h) < 0$ when h is small enough.

3. PROOFS

Proof of Theorem 1: We know that the operators P_h can also be defined using the optimality system. Consider the optimality system corresponding to problem (1.5), (1.6):

$$(3.1)\quad \begin{cases} x_h^{*\prime}(t) = A_h x_h^*(t) + B_h B_h^* p_h(t), t > 0 \\ p'_h(t) = -A_h^* p_h(t) + C_h^* C_h x_h^*(t), t > 0 \\ x_h^*(0) = x_0, \ \lim_{t\to\infty} p_h(t) = 0 \end{cases}$$

then P_h can be defined by:

$$P_h x_0 = -p_h(0), \text{ for all } x_0 \in X.$$

We know that (3.1) has a unique solution (x_h^*, p_h), $p_h \in L^2(0, +\infty; X)$ and:

$$(3.2)\quad p_h(t) = -P_h x_h^*(t), \text{ for all } t \geq 0$$

Let x_h be the solution to the system:

$$x'_h(t) = (A_h + B_h F) x_h(t), \ t > 0, \ x_h(0) = x_0,$$

or, if we use the notation in (iv)

$$x_h(t) = S_{h,F}(t) x_0, \ t \geq 0.$$

Then by (iv)(1.10) we deduce that $J_h(Fx_h, x_h) < +\infty$. From (ii)(1.9) and the uniform boundedness principle, we have that $\|C_h\| \leq c$ for all h and using now (1.14) and (iv)(1.10) we obtain that, for all h:

$$(P_h x_0, x_0) \leq \widetilde{M}\|x_0\|^2, \ x_0 \in X$$

Thus we have:

$$(3.3)\quad \|P_h\| \leq \widetilde{M} \text{ for all } h.$$

Relations (3.3) and (1.14) yield that the sequences $\{u_h^*\}$ and $\{C_h x_h^*\}$ are bounded in $L^2(0, +\infty; U)$ and $L^2(0, +\infty; Z)$ respectively. Hence, one can find $\overline{u} \in L^2(0, +\infty; U)$ such that:

$$(3.4)\quad u_h^* \to \overline{u} \text{ weakly in } L^2(0, +\infty; U) \text{ as } h \text{ A } 0$$

and also

(3.5) $$x_h^*(t) \to \bar{x}(t) \text{ weakly in } X \text{ as } h \text{ A } 0, \text{ for each } t \geq 0$$

where $\bar{x}$ is the mild solution corresponding to $\bar{u}$ to the system (1.3). $\bar{x}$ satisfies also:

(3.6) $$C_h x_h^* \to C\bar{x} \text{ weakly in } L^2(0, +\infty; Z) \text{ as } h \text{ A } 0.$$

From (3.2) and (3.5) we deduce that the sequence $\{p_h(t)\}$ is bounded in X for each $t \geq 0$, and so:

(3.7) $$p_h(t) \to \bar{p}(t) \text{ weakly in } X \text{ as } h \text{ A } 0, \text{ for each } t \geq 0,$$

where $\bar{p}$ satisfies:

$$\bar{p}'(t) = -A^* \bar{p}(t) + C^* C\bar{x}(t), \quad t > 0$$

i.e., for some $T > 0$ we have:

$$\bar{p}(t) = S^*(T-t)\bar{p}(T) - \int_t^T S^*(s-t)C^*C\bar{x}(s)\,ds, \quad 0 < t < T.$$

We denote by $\{S(t);\, t \geq 0\}$ and $\{S_h(t); t \geq 0\}$ the semigroups generated by A and A_h respectively.

From (3.1) we have that, for T > 0 arbitrary:

$$p_h(t) = S_h^*(T-t)p_h(T) - \int_t^T S_h^*(s-t)C_h^*C_h x_h^*(s)\,ds, \text{ for } t > 0,$$

Letting T → ∞ in the above relation, and knowing that $\lim_{t\to\infty} p_h(t) = 0$ we obtain:

$$p_h(t) = -\int_t^\infty S_h^*(s-t)C_h^*C_h x_h^*(s)\,ds.$$

We also know that:

$$u_h^*(t) = B_h^* p_h(t), \quad t \geq 0,$$

so, one can consider p_h as the solution to the following system:

$$p'_h(t) = -(A_h^* + B_h F)^* p_h(t) + C_h^* C_h x_h^*(t) + F^* u_h^*(t),$$

i.e.,

(3.8) $$p_h(t) = -\int_t^\infty S_{h,F}^*(s-t)\left(C_h^*C_h x_h^*(s) + F^* u_h^*(s)\right)ds.$$

Using (1.10), (ii)(1.9), (3.6), (3.4) and the uniqueness of the weak-limit, letting h A 0 in (3.8) we deduce that:

$$\bar{p}(t) = -\int_t^\infty S_F^*(s-t)\left(C^*C\bar{x}(s) + F^*\bar{u}(s)\right)ds,$$

and by (1.11), $\bar{p} \in L^2(0, +\infty; X)$. Thus, the pair $(\bar{x}, \bar{p})$ satisfies the optimality system:

$$\begin{cases} \bar{x}'(t) = A\bar{x}(t) + BB^*\bar{p}(t), \\ \bar{p}'(t) = -A^*\bar{p}(t) + C^*C\bar{x}(t), \\ \bar{x}(0) = x_0, \lim_{t\to\infty} \bar{p}(t) = 0. \end{cases}$$

The solution to this system being unique, we have that $(\bar{x}, \bar{u})$ is the optimal pair (x^*, u^*) for the problem (1.4), (1.3).

Hence $\bar{p}(t) = -Px^*(t)$ where P is the linear, bounded, selfadjoint, positive solution to (1.2). By (3.7) we have (2.1).

Proof of the Lemma. We shall prove that, for h small enough, we have the following inclusion:

(3.9) $$\rho(A_1) \subseteq \rho(A_1 + T_h).$$

Taking $\lambda \in \rho(A_1)$, we must show that the equation:

(3.10) $$\lambda x - A_1 x - T_h x = f$$

has a unique solution for each $f \in X$, in order to have (3.9). If we put $y = \lambda x - A_1 x$, (3.10) is equivalent to:

(3.11) $$y - T_h R(\lambda; A_1) y = f.$$

From (a) we deduce that $R(\lambda; A_1)$ is a compact operator, for all $\lambda \in \rho(A_1)$, not only for $\lambda = \lambda_0$, and by (b) we have that the operator $T_h R(\lambda; A_1)$ is also compact, for all h and $\lambda \in \rho(A_1)$.

Thus, to prove that (3.11) has a unique solution for each $f \in X$ one can use the Fredholm alternative, which says that (3.11) has a unique solution if and only if the equation:

(3.12) $$z - R(\lambda; A_1^*)T_h^* z = 0$$

has only the trivial solution $z_h = 0$.

Suppose by absurd, that there exists $z_h \in X$, $\|z_h\| = 1$ such that:

(3.13) $$z_h - R(\lambda; A_1^*)T_h^* z_h = 0$$

The sequence $\{z_h\}$ being bounded $(\|z_h\| = 1)$ it is weakly convergent:

(3.14) $$z_h \to z, \text{ weakly in } X \text{ as } h \searrow 0.$$

Because $R(\lambda; A_1)$ is compact and $\{T_h^* z_h\}$ bounded, we have:

(3.15) $$R(\lambda; A_1)T_h^* z_h \to \bar{z}, \text{ strongly in } X \text{ as } h \searrow 0.$$

By (3.13), (3.14), (3.15) and the uniqueness of the limit of a sequence, we deduce that $z = \tilde{z}$ and that the convergence in (3.14) is, in fact, in the strong topology of X. By (b), we have:

$$\left(T_h^* z_h, w\right) = \left(z_h, T_h w\right) \to 0 \text{ as } h \searrow 0, \text{ for all } w \in X.$$

From this last relation and (3.15) we deduce that $z = \tilde{z} = 0$ which is in contradiction with $\|z_h\| = 1$ and the strong convergence of $\{z_h\}$ to z.
Hence (3.9) is true. From (3.9) we deduce that $\sigma(A_1 + T_h) \subseteq \sigma(A_1)$, for h small enough, and so $s(A_1 + T_h) \le s(A_1) < 0$.

Proof of Theorem 2. Let $\{S_h(t);\ t \ge 0\}$ be the semigroup generated by the operator $A_1 + T_h$. In order to prove the theorem we shall prove that $\omega_0(S_h) < 0$, which by the s.d.g. condition 1 is equivalent to $s(A_1 + T_h) < 0$, for h small enough. If $\{S^1(t); t \ge 0\}$ is the semigroup generated by A_1, then assumption 2 implies that $\omega_0(S^1) < 0$ and, because $s(A_1) \le \omega_0(S^1)$ we also have $s(A_1) < 0$. Because of 3 and 4 we can use the Lemma to deduce that:

$$\omega_0(S_h) = s(A_1 + T_h) < 0, \text{ for } h \text{ small enough}$$

which concludes the proof.

4. APPLICATION TO DELAY EQUATIONS

Consider the problem of minimizing (1.4) where the state is given by the following delay equation:

$$\text{(4.1)} \qquad \begin{cases} z'(t) = A_1 z(t) + A_2 z(t-r) + B_0 u(t), \ t > 0 \\ z(0) = h_0, \ z(\theta) = h_1(\theta) \ \text{for } -r \le \theta \le 0, \end{cases}$$

where $h_0 \in \mathbf{R}^n$, $h_1 \in L^2(-r, 0; \mathbf{R}^n)$ are given, r is a given positive constant-the delay, and A_1, A_2, B_0 are real matrices, $A_1, A_2 \in \mathbf{R}^{n \times n}$, $B_0 \in \mathbf{R}^{m \times n}$.
Equation (4.1) can be written in the abstract form (1.3) as follows (see [3]): define $X = \mathbf{R}^n \times L^2(-r, 0; \mathbf{R}^n)$, $U = \mathbf{R}^m$, and $Z = \mathbf{R}^p$.
The operator $\mathscr{A}$: $D(\mathscr{A}) \subseteq X \to X$ is given by:

$$\text{(4.2)} \qquad D(\mathscr{A}) = \{(h_0, h_1) \in X;\ h_1 \in W^{1,2}(-r, 0; \mathbf{R}^n),\ h_1(0) = h_0\},$$

$$\text{(4.3)} \qquad \mathscr{A}(h_0, h_1) = (A_1 h_0 + A_2 h_1(-r),\ h'_1).$$

The semigroup generated by $\mathscr{A}$ is given by:

$$\mathscr{S}(t): X \to X, \quad \mathscr{S}(t)(h_0, h_1) = (z(t), z_t)$$

where z is the solution to (4.1) corresponding to (h_0, h_1) and z_t is the function defined by:

$$z_t(\theta) = z(t + \theta),\ -r \le \theta \le 0.$$

The semigroup $\mathscr{S}(t)$ is differentiable for $t \ge r$ (see[3]). With $\Delta(\lambda) = \lambda I - A_1 - \exp(-\lambda r)A_2$, the spectrum of $\mathscr{A}$ is given by:

$$\sigma(\mathscr{A}) = \{\lambda \in \mathbf{C};\ \det \Delta(\lambda) = 0\}.$$

The operator $\mathscr{B}$: $U \to X$ is given by:

$$\text{(4.4)} \qquad \mathscr{B}u = (B_0 u, 0),\ \text{for all } u \in U.$$

and it is compact.
The observation operators $\mathscr{C}, \mathscr{C}^N: X \to Z$ are given by:

$$\text{(4.5)} \qquad \mathscr{C}(h_0, h_1) = C_0 h_0,\ \mathscr{C}^N(h_0, h_1) = C_0^N h_0 \ \text{for all } (h_0, h_1) \in X$$

where the real matrices $C_0^N, C_0 \in \mathbf{R}^{n \times p}$ are chosen such that (ii) holds.

We shall present the averaging approximation of delay equations given in [1].

For each integer N, we divide the interval $[-r,0]$ into N subintervals $\left[t_j^N, t_{j-1}^N\right]$, $j = \overline{1,N}$, where $t_j = -jr/N$. Let χ_j^N denote the characteristic function of $\left[t_j^N, t_{j-1}^N\right]$ for $j = \overline{2,N}$ and χ_1^N the characteristic function of $\left[t_1^N, t_0^N\right] = [-r/N, 0]$.

Consider the finite dimensional space:

$$\text{(4.6)} \qquad X^N = \left\{(h_0, h_1) \in X;\ h_1 = \sum_{j=1}^{N} v_j^n \chi_j^n,\ v_j^N \in \mathbf{R}^n,\ j = \overline{1,N}\right\},$$

and the operator $\mathscr{A}^N: X \to X^N$ defined as:

$$\text{(4.7)} \qquad \mathscr{A}^N(h_0, h_1) = \left(A_1 h_0^N + A_2 h_N^N,\ \sum_{j=1}^{N} \frac{N}{r}\left(h_{j-1}^N - h_j^N\right)\chi_j^N\right),$$

where

$$\text{(4.8)} \qquad h_0^N = h_0,\ h_j^N = \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N} h(\theta)\,d\theta,\ j = \overline{1,N}.$$

Obviously, the parameter h is $1/N$. We do not need to approximate $\mathscr{B}$, $\mathscr{B}^N = \mathscr{B}$ for all N.

LEMMAS 3.6, 3.2 from [1] and Theorem 4.5 from [11] ensure that (i) is satisfied.

We suppose that the pairs $(\mathscr{A}, \mathscr{C})$, $(\mathscr{A}^N, \mathscr{C}^N)$ are detectable, for all N. The pair $(\mathscr{A}, \mathscr{C})$ is detectable if and only if (see [10]):

$$rank\,(\Delta(\lambda)^T,\ C_0^T) = n \ \text{for all } \lambda \in \mathbf{C},\ \mathscr{R}\,\lambda \ge 0.$$

We suppose that the pair $(\mathscr{A}, \mathscr{B})$ is stabilizable, i.e., there exists $\mathscr{F} \in \mathscr{L}(X, U)$ such that $\mathscr{A} + \mathscr{B}\mathscr{F}$ generates an exponentially stable semigroup. A necessary and sufficient condition for the stabilizability of the pair $(\mathscr{A}, \mathscr{B})$ is ([10]):

$$rank(\Delta(\lambda), B_0) = n \text{ for all } \lambda \in \mathbf{C},\ \mathcal{R}\lambda \geq 0.$$

In order to have (iii) we must prove that $\mathcal{A}^N + \mathcal{B}\mathcal{F}$ generate exponentially stable semigroups, for $N \geq N_0$. $\mathcal{A}^N + \mathcal{B}\mathcal{F}$ is a linear bounded operator, $\mathcal{A} + \mathcal{B}\mathcal{F}$ generates a semigroup $\mathcal{S}_F(t)$ for which there exists $t_0 > 0$, s.t. $\mathcal{S}_F(t_0)$ is compact, and we conclude that $\mathcal{A} + \mathcal{B}\mathcal{F}$ and $\mathcal{A}^N + \mathcal{B}\mathcal{F}$ satisfy the s.d.g. condition. In order to show (1.10) we may try to prove that $s(\mathcal{A} + \mathcal{B}\mathcal{F}) < 0$. Adapting the proof of Lemma 3.4 in [1] we can show that:

$$s(\mathcal{A}^N + \mathcal{B}\mathcal{F}) \to s(\mathcal{A} + \mathcal{B}\mathcal{F}) \text{ as } N \to \infty. \tag{4.9}$$

But $s(\mathcal{A}+\mathcal{B}\mathcal{F}) < 0$ which implies that $s(\mathcal{A}^N+\mathcal{B}\mathcal{F}) < 0$ for $N \geq N_0$.

Using the Arzelà-Ascoli theorem one can easily prove that the assumption (α) in Corollary is true. The condition (β) is fulfilled because the semigroup generated by $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}^N$ has a compact element. From the Corollary we deduce that $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{P}^N$ generates an exponentially stable semigroup, if $N \geq N_0$.

From Theorem 1 we deduce only the weak convergence:

$$\mathcal{P}^N(h_0,h_1) \to \mathcal{P}(h_0,h_1) \text{ when } N \to \infty, \text{ for all } (h_0, h_1) \in X,$$

but, in fact, the convergence is in the strong topology of X. This problem was studied in many papers (e.g. [7], [13], [6], [5], [8]) under stronger hypotheses on the operators involved, assumptions which are satisfied by the quadratic control problem with state given by a delay equation presented above.

We expect to give numerical results in a later paper.

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