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A SEQUENCE OF POSITIVE LINEAR OPERATORS

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1. Let $a > 0$ be a real number. For $n \geq 1$ let $L_n : C[-a-1, a+1] \to C[-a, a]$,

$$L_n f(x) := n![x + h_0, \ldots, x + h_n; F_n],$$

where $f \in C[-a-1, a+1]$, $x \in [-a, a]$, $h_i = -1 + \frac{2i}{n}$, $i = 0, \ldots, n$, $F_n \in C^n[-a-1, a+1]$, $F_n^{(n)} = f$.

For convex functions f, various inequalities involving $L_n f$ have been studied in [3], [4], [5], [12].

We have also

$$L_n f(x) = 2^{-n} \int_{x-1}^{x+1} \cdots \int_{x-1}^{x+1} f\left(\frac{t_1 + \cdots + t_n}{n}\right) \mathrm{d}t_1 \ldots \mathrm{d}t_n$$

As positive operators, L_n have been studied in [6], [9].

In this paper we present a Voronovskaya type result for the operators L_n.

2. Consider the polynomial

$$(X - h_0) \cdots (X - h_n),$$

which we write

$$X^{n+1} - c_1 X^n - \cdots - c_n X - c_{n+1}.$$

Consider the sum

$$S_p := \sum_{i=0}^{n} h_i^p, \quad p = 1, 2, \ldots$$

Clearly

(1) $$S_p = 0, \quad \text{for odd } p.$$

We have

$$\lim_{n\to\infty}\frac{S_p}{n} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n} h_i^p = \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\left(\frac{2i}{n}-1\right)^p =$$

$$= \int_0^1 (2x-1)^p \mathrm{d}x = \frac{1}{p+1}, \text{ for even } p.$$

Denoting by 1_n the element of a sequence converging to 1, we obtain

$$S_{2q+1} = 0$$

(2) $$S_{2q} = \frac{n}{2q+1}1_n, \ q = 0,1,\ldots$$

We have calculated the following S_p numbers:

$$S_2 = \frac{n}{3} + \frac{2}{3n} + 1,$$

$$S_4 = \frac{n}{5} + \frac{4}{3n} - \frac{8}{15n^3} + 1,$$

$$S_6 = \frac{n}{7} + \frac{2}{n} - \frac{8}{3n^3} + \frac{32}{21n^5} + 1.$$

In order to calculate the coefficients

$$c_p = (-1)^{p+1}\sum_{0\le i_1<\cdots<i_p\le n} h_{i_1}\ldots h_{i_p},$$

we shall use the Newton formulas:

$$c_1 = S_1$$

$$c_p = \frac{1}{p}\left(S_p - \sum_{i=1}^{p-1} S_i c_{p-i}\right) \quad p = 2,\ldots,n+1.$$

Taking into account (2) we obtain:

$$c_p = 0, \text{ for odd } p,$$

and

$$c_2 = \frac{1}{2}S_2$$

(3) $$c_{2q} = \frac{1}{2q}\left(S_{2q} - \sum_{i=1}^{q-1} S_{2i} c_{2(q-i)}\right) \quad q = 1,\ldots,[(n+1)/2].$$

We have calculated the following c_i values:

$$c_2 = \frac{n}{6} + \frac{1}{3n} + \frac{1}{2},$$

$$c_4 = -\frac{n^2}{72} - \frac{n}{30} + \frac{1}{6n} - \frac{1}{18n^2} - \frac{2}{15n^3} + \frac{5}{72},$$

$$c_6 = \frac{n^3}{1296} - \frac{n^2}{720} - \frac{263n}{15120} + \frac{29}{216n} - \frac{11}{180n^2} - \frac{301}{810n^3} + \frac{2}{45n^4} + \frac{16}{63n^5} + \frac{13}{720}.$$

Let

$$\gamma_q := \lim_{n\to\infty}\frac{c_{2q}}{n^q}, \ q \ge 1.$$

We have

$$\gamma_1 = \frac{1}{6},$$

and, from (3), it follows

$$\gamma_q = -\frac{\gamma_{q-1}}{6q}, \ q \ge 2,$$

hence

(4) $$\gamma_q = \frac{(-1)^{q+1}}{6^q q!}, \ q \ge 1.$$

Consider the functions $e_i : [-1,1] \to \mathbb{R}$, $e_i(t) = t^i$, $i = 0,1,\ldots$. Using the divided difference functional, define the numbers:

$$\alpha_j = [h_0,\ldots,h_n; e_{n+j}], \ j = -n,-n+1,\ldots$$

It is well known that

(5) $$\alpha_j = \begin{cases} 0, & \text{if } j = -n,\ldots,-1, \\ 1, & \text{if } j = 0. \end{cases}$$

In order to calculate α_j for $j \ge 1$, observe that

$$[h_0, \ldots, h_n; e_{j-1}(e_1 - h_0e_0)\ldots(e_1 - h_ne_0)] = 0,$$

that is

$$[h_0, \ldots, h_n; e_{n+j} - c_1e_{n+j-1} - \ldots - c_{n+1}e_{j-1})] = 0,$$

We obtain

$$\alpha_j = \sum_{k=1}^{n+1} c_k \alpha_{j-k}, \ j = 1,2,\ldots,$$

and using (5), we find that

(6)
$$\alpha_j = \sum_{k=1}^{j} c_k \alpha_{j-k}, \quad j = 1, \ldots, n+1.$$

Using

$$\alpha_0 = 1$$
$$\alpha_1 = c_1 \alpha_0 = 0$$

in (6), we obtain

(7)
$$\alpha_p = 0, \quad \text{for odd } p,$$

and hence

(8)
$$\alpha_{2q} = \sum_{i=1}^{q} c_{2i} \alpha_{2(q-i)}, \quad 1 \le q \le \frac{n+1}{2}.$$

Define

$$\beta_q := \lim_{n \to \infty} \frac{\alpha_{2q}}{n^q}, \quad q \ge 0.$$

Using (8) we find

$$\beta_q = \sum_{i=1}^{q} \gamma_i \beta_{q-i}, \quad q \ge 1,$$

i.e.

$$\beta_q = \sum_{i=1}^{q} \frac{(-1)^{i+1}}{6^i\, i!} \beta_{q-i}, \quad q \ge 1,$$

which implies

$$\beta_q = \frac{1}{6^q\, q!}, \quad q \ge 0,$$

hence

(9)
$$\alpha_{2q} = \frac{n^q}{6^q\, q!} 1_n, \quad q \ge 0.$$

We have calculated the following α_j values:

$$\alpha_2 = \frac{n}{6} + \frac{1}{3n} + \frac{1}{2},$$

$$\alpha_4 = \frac{n^2}{72} + \frac{2n}{15} + \frac{1}{2n} + \frac{1}{18n^2} - \frac{2}{15n^3} + \frac{31}{72},$$

$$\alpha_6 = \frac{n^3}{1296} + \frac{11n^2}{720} + \frac{1753n}{15120} + \frac{137}{216n} + \frac{7}{60n^2} - \frac{409}{810n^3} - \frac{2}{45n^4} + \frac{16}{63n^5} + \frac{33}{80}.$$

3. Observe that

$$L_n(f)(x) := n!\left[h_0, \ldots, h_n; F_n \circ (e_1 + xe_0)\right].$$

Hence we obtain

$$L_n\left((e_1 - xe_0)^j\right)(x) = \frac{n!\, j!}{(n+j)!}\left[h_0, \ldots, h_n; e_1^{n+j}\right] = \frac{n!\, j!}{(n+j)!}\left[h_0, \ldots, h_n; e_{n+j}\right] = \frac{n!\, j!}{(n+j)!} \alpha_j,$$

$j = 1, 2, \ldots$

Using (7) and (9), it follows

$$L_n\left((e_1 - xe_0)^{2q+1}\right)(x) = 0,$$

(10)
$$L_n\left((e_1 - xe_0)^{2q}\right)(x) = \frac{(2q)!}{6^q\, q!} n^{-q} 1_n, \quad q \ge 0.$$

We shall use the following values:

$$L_n\left((e_1 - xe_0)^2\right)(x) = \frac{1}{3n},$$

(11)
$$L_n\left((e_1 - xe_0)^4\right)(x) = \frac{1}{3n^2} - \frac{2}{15n^3},$$

$$L_n\left((e_1 - xe_0)^6\right)(x) = \frac{5}{9n^3} - \frac{2}{3n^4} + \frac{16}{63n^5}.$$

4. Combining (10) and [11, Cor.2] we obtain

(12)
$$\lim_{n \to \infty} n^k \left(L_n f(x) - \sum_{q=0}^{k-1} \frac{L_n (e_1 - xe_0)^{2q}(x)}{(2q)!} f^{(2q)}(x) \right) = \frac{f^{(2k)}(x)}{6^k\, k!}, \quad k = 0, 1, \ldots$$

provided f is $2k$ times differentiable at x.

For $k = 0$ we have $\lim_{n\to\infty} L_n f(x) = f(x)$; related results are to be found in [1, 3.30], [2], [8], [10, 6.8.1].

For $k = 1$, (12) yields

$$\lim_{n \to \infty} n\left(L_n f(x) - f(x)\right) = \frac{1}{6} f''(x); \text{ see also [1, 3.32].}$$

For $k = 3$ we infer

$$\lim_{n\to\infty} n\left(n\left(n\left(L_n f(x)-f(x)\right)-\frac{f''(x)}{6}\right)-\frac{f^{IV}(x)}{72}\right)=\frac{f^{VI}(x)}{1296}-\frac{f^{IV}(x)}{180}.$$

Remark. Other results concerning the approximation of a divided difference by derivatives are to be found in [7].

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