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A QUASI-OPTIMUM JOR METHOD

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1. INTRODUCTION

In this paper we present a result on the Jacobi overrelaxation (JOR) method. This result is discussed under some restrictions on the spectrum of the matrix A, for a system of linear equations $Ax = b$, where $A \in \mathbb{C}^{n,n}$ is a nonsingular matrix with nonzero diagonal entries and $x, b \in \mathbb{C}^n$. Without loss of generality we can suppose that $a_{ii} = 1$, $i = 1,2,\dots,n$. The associated JOR method which has been introduced by Young [1], can be written as

$$x^{n+1} = Mx^n + d, \quad x^0 \in \mathbb{C}^n$$

where $M = I - rA$, $d = rb$, $r \in \mathbb{R}$, $r \neq 0$. In this paper we give the JOR method with a quasi-optimum parameter r. This method is particularly useful if the JOR optimum parameter cannot be determined or if the eigenvalues of the matrix A can be only approximately determined.

2. A CHOICE OF r

THEOREM 1. *If all eigenvalues of the matrix A are in a circle whose limit circumference has the following properties: the center of the circumference is on the real axis, the set $\{(T,0),(t,0)\}$ is intersection of the circumference and the real axis, where either $T \geq 3t > 0$ or $T \leq 3t < 0$.*

If $r = 4|t|/(4t^2 + (T-t)^2)$ then for spectral radius $p(M)$ of the JOR iteration matrix M it holds

$$p(M) \leq |T - t| \,/\, ((T-t)^2 + 4t^2)^{1/2} < 1$$

Proof. The case $T \geq 3t > 0$. For the spectral radius $p(M)$ it holds $p(M) = \max\limits_{1 \leq i \leq n} |1 - rz_i|$ where z_i are the eigenvalues of the matrix A. Let $z_k = a_k + ib_k$,

$a_k, b_k \in \mathbb{R}$, $k = 1,2,\ldots,n$, then $p(M) = \max_{1\le k\le n}\left((1-ra_k)^2 + r^2 b_k^2\right)^{1/2}$. Since $T\ge 3t>0$ hence $T>t>0$. Now $(T-3t)(T-t)\ge 0$, $(4t^2+(T-t)^2)/2t\ge T+t$ and $2/r\ge T+t$. Since $t\le a_k\le T$, $k=1,2,\ldots,n$ hence $2/r\ge T+t\ge a_k+t$, i.e. $2/r\ge a_k+t$. Now $2(a_k-t)/r\ge (a_k+t)(a_k-t)$, $(1-rt)^2\ge (1-ra_k)^2$. For the numbers b_k $k=1,2,\ldots,n$ it holds $|b_k|\le (T-t)/2$. For the spectral radius $p(M)$ it holds $p(M)\le((1-rt)^2+r^2(T-t)/4)^{1/2}=f(r)$ Since $f'(4t/(4t^2+(T-t)^2)) = 0$ $f''(4t/(4t^2+(T-t)^2)) > 0$ we have $\min f(r) = f(4t/(4t^2+(T-t)^2)) = (T-t)/(4t^2+(T-t)^2)^{1/2} < 1$. If $T\ge 3t>0$ then we consider the system $-Ax=-b$ and we have $p(M) = \max_{1\le i\le n}|1+rz_i|$ where z_i are the eigenvalues of the matrix A. Similarly to the proof of the precedent we have $(1+ra_k)^2\le(1+rt)^2$. The rest of the proof follows similarly to the proof of the precedent.

THEOREM 2. *If all eigenvalues of the matrix A are in a circle whose limit circumference has the following properties: the center of the circumference is on the real axis, the set* $\{(T,0),(t,0)\}$ *is intersection of the circumference and the real axis where either* $T\ge t>0$ *or* $T\le t<0$.

If $r = |t|/T^2$ *then for spectral radius* $p(M)$ *of the matrix M it holds* $p(M)\le(T^2-t^2)^{1/2}/|T|<1$.

Proof. The case $T\ge t>0$. Similarly to the proof of the Theorem 1. we have $p(M) = \max_{1\le i\le n}((1-ra_k)^2+r^2b_k^2)^{1/2}$

For the numbers b_k $k=1,2,\ldots,n$ it holds

$$b_k^2 \le \left(\frac{T-t}{2}\right)^2 - \left(\frac{T+t}{2}-a_k\right)^k. \text{ Now}$$

$$p(M)\le\max_{1\le k\le n}\left(1-2ra_k+r^2a_k^2+r^2\left(\left(\frac{T-t}{2}\right)^2-\left(\frac{T+t}{2}\right)^2+a_k(T+t)-a_k^2\right)\right)^{1/2}$$

Since $t\le a_k\le T$ we have $p(M)\le((1-rt)^2+r^2(T^2-t^2))^{1/2}=f(r)$

Now $f'(t/T^2)=0$, $\quad f''(t/T^2)>0$

$$\min f(r) = f(t/T^2) = (T^2-t^2)^{1/2}/T < 1$$

If $T\le t<0$ then we consider the system $-Ax=-b$. Similarly to the proof of the precedent we have

$p(M)\le((1+rt)^2+r^2(T^2-t^2))^{1/2}=f(r)$. The rest of the proof follows similarly to the proof of the precedent.

Theorems 1 and 2 give the upper bounds for the spectral radius $p(M)$ of the matrix M.

Since $\dfrac{|T-t|}{\left((T-t)^2+4t^2\right)^{1/2}} < \dfrac{(T^2-t^2)^{1/2}}{|T|}$ for $T\ge t>0$ or $T\le t<0$ we give the following conclusion:

if $0<t\le T<3t$ or $3t<T\le t<0$ we can use Theorem 2.

if $T\ge 3t>0$ or $T\le 3t<0$ we can use Theorem 1.

THEOREM 3. *If the matrix A has the only one eigenvalue z, then it holds* $p(M)=0$.

Proof. Let $z_1, z_2, \ldots, z_n$ be the eigenvalues of the matrix A. Then it holds

$$\sum_{i=1}^{n} z_i = nz = \sum_{i=1}^{n} a_{ii} = n$$

and we have $z=1$, $T=t=1$. Now from Theorems 1 and 2 there follows $p(M)=0$.

3. EXAMPLE

Let

$$A = \begin{bmatrix} 1 & 0.5 & -0.5 \\ 0.5 & 1 & -0.5 \\ 0.5 & -0.5 & 1 \end{bmatrix}$$

The eigenvalues of A are $z = 0.5$, $z_2 = 1$, $z_3 = 1.5$. The best choice for the JOR method is $r = 1$ i.e. the Jacobi method which yields $p(M) = 0.5$. From Theorem 1 we obtain $r = 1$ i.e. the Jacobi method and $p(M)\le 0.707$. From Theorem 2 we obtain $r = 2/9$ and $p(M)\le 0.94$.

REFERENCE

1. Young D.M., *Iterative solution of large linear systems*, Academic Press, New York/London, 1971.

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