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AN IDENTITY FOR ULTRASPHERICAL POLYNOMIALS

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1. The aim of this paper is to give an identity for ultraspherical polynomials and some applications. We need to introduce Jacobi polynomials $P_n^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$. These polynomials can be defined, for example, by the hypergeometric representation

$$(1)\qquad P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right),$$

where

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!},\ |z| < 1.$$

The shifted factorial $(a)_k$ is given by

$$(a)_0 = 1$$

and

$$(a)_k = a(a+1)\dots(a+k-1),\ k \geq 1.$$

If $\alpha > -1, \beta > -1$, then the Jacobi polynomials are orthogonal with respect to the weight function $\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on the interval (-1, 1).

Some special cases of Jacobi polynomials (1), except a constant factor, are:
- the Tchebycheff polynomials of the first kind

$$T_n(x) = \cos n\theta,\ x = \cos\theta,\ \text{for } \alpha = \beta = -\frac{1}{2};$$

- the Tchebycheff polynomials of the second kind

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},\ x = \cos\theta,\ \text{for } \alpha = \beta = \frac{1}{2};$$

- the Legendre polynomials for $\alpha=\beta=0$;
- the ultraspherical polynomials for $\alpha=\beta$.

2. Let us denote by $R_n^{(\alpha,\beta)}$ the polynomials

$$R_n^{(\alpha,\beta)}(x)=\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} \tag{2}$$

where

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}=\frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}. \tag{3}$$

Further properties, of $P_n^{(\alpha,\beta)}$ are

$$P_n^{(\alpha,\beta)}(x)=(-1)^n P_n^{(\beta,\alpha)}(-x) \tag{4}$$

and consequently

$$P_n^{(\alpha,\beta)}(-1)=(-1)^n\binom{n+\beta}{n}=(-1)^n\frac{\Gamma(n+\beta+1)}{n!\,\Gamma(\beta+1)}. \tag{5}$$

THEOREM 1. *The following identity*

$$(1+x)^{\alpha}\sum_{k=0}^{n}R_k^{(\alpha,\alpha)}(x)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-1}^{x}(1+y)^{\alpha-\frac{1}{2}}(x-y)^{-\frac{1}{2}}\frac{1-R_{n+1}^{\left(\alpha-\frac{1}{2};\alpha-\frac{1}{2}\right)}(y)}{1-y}dy \tag{6}$$

is true for $\alpha\geq 0$ *and every* $x\in(-1,\,1)$.

Proof. We consider the integral representation (see [1]–[3])

$$(1+x)^{b+\mu}\frac{P_k^{(a-\mu,b+\mu)}(x)}{P_k^{(a-\mu,b+\mu)}(-1)}=\frac{\Gamma(b+\mu+1)}{\Gamma(b+1)\Gamma(\mu)}\int_{-1}^{x}(1+y)^{b}(x-y)^{\mu-1}\frac{P_k^{(a;b)}(y)}{P_k^{(a;b)}(-1)}dy,$$

$x\in(-1,1),\mu>0,b>-1.$

According to (5) we may write

$$(1+x)^{b+\mu}R_k^{(a-\mu,b+\mu)}(x)\frac{\binom{k+a-\mu}{k}}{\binom{k+b+\mu}{k}}=\frac{\Gamma(b+\mu+1)}{\Gamma(b+1)\Gamma(\mu)}\int_{-1}^{x}(1+y)^{b}(x-y)^{\mu-1}R_k^{(a,b)}(y)\frac{\binom{k+a}{k}}{\binom{k+b}{k}}dy$$

Using the equality

$$R_k^{(\alpha+1,\beta)}(x)=\frac{2(\alpha+1)}{2k+\alpha+\beta+2}\cdot\frac{R_k^{(\alpha,\beta)}(x)-R_{k+1}^{(\alpha,\beta)}(x)}{1-x},$$

it follows for $a-\mu=b+\mu=\alpha$ that

$$(1+x)^{\alpha}R_k^{(\alpha,\alpha)}(x)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)\Gamma(\mu)}\int_{-1}^{x}(1-y)^{\alpha-\mu}(x-y)^{\mu-1}R_k^{(\alpha+\mu,\alpha-\mu)}(y)\frac{\binom{k+\alpha+\mu}{k}}{\binom{k+\alpha-\mu}{k}}dy=$$

$$=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)\Gamma(\mu)}\int_{-1}^{x}(1-y)^{\alpha-\mu}(x-y)^{\mu-1}\frac{R_k^{(\alpha+\mu-1,\alpha-\mu)}(y)-R_{k+1}^{(\alpha+\mu-1,\alpha-\mu)}(y)}{1-y}C(k,\alpha,\mu)dy,$$

where

$$C(k,\alpha,\mu)=\frac{2(\alpha+\mu)}{2\alpha+2k+1}\cdot\frac{\Gamma(k+\alpha+\mu+1)}{\Gamma(\alpha+\mu+1)}\cdot\frac{\Gamma(\alpha-\mu+1)}{\Gamma(k+\alpha-\mu+1)}$$

The last equality implies, for $\mu=\frac{1}{2}$, that

$$(1+x)^{\alpha}R_k^{(\alpha,\alpha)}(x)=\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-1}^{x}(1+y)^{\alpha-\frac{1}{2}}(x-y)^{-\frac{1}{2}}\frac{R_k^{\left(\alpha-\frac{1}{2};\alpha-\frac{1}{2}\right)}(y)-R_{k+1}^{\left(\alpha-\frac{1}{2};\alpha-\frac{1}{2}\right)}(y)}{1-y}dy$$

for $x\in(-1,\,1)$, $k=0,1,2,\ldots$

By summation, for $k=0,1,\ldots,n,$ results the required identiy (6).

3. We can use the identity (6) to obtain a new proof of an inequality for ultraspherical polynomials (see also [1]-[3]).

COROLLARY 1. *If* $\alpha\geq 0$ *then the inequality*

$$\sum_{k=0}^{n}R_k^{(\alpha,\alpha)}(x)>0 \tag{7}$$

holds for every $x\in(-1,\,1)$ *and* $n=0,1,2,\ldots$.

Proof. For $x \in (-1, 1)$, and $\alpha \geq 0$, the inequality

$$\left| R_{n+1}^{\left(\alpha-\frac{1}{2};\alpha-\frac{1}{2}\right)}(x) \right| < 1$$

is valid [4]. This implies that the right-side of (6) is positive. This completes the proof.

COROLLARY 2. *Let p_n be a polynomial of degree n represented by*

$$p_n(x) = \sum_{k=0}^{n} \alpha_k R_k^{(\alpha,\alpha)}(x)$$

for $x \in (-1, 1)$ and α_k real coefficients, $k=0,1,\ldots,n$. If $\alpha_0 \geq \alpha_1 \geq \ldots \alpha_n > 0$ then $p_n(x) > 0$ for every $x \in (-1, 1)$.

Proof. Using the identity of Abel we have

$$p_n(x) = \sum_{k=0}^{n-1} (\alpha_k - \alpha_{k+1}) \sum_{j=0}^{k} R_j^{(\alpha,\alpha)}(x) + \alpha_n \sum_{j=0}^{n} R_j^{(\alpha,\alpha)}(x).$$

From the monotony of the sequence α_k, $k=0,1,\ldots,n$, and the inequality (7) results that

$$p_n(x) > 0, \text{ for } x \in (-1,1).$$

Another application may be described in the following way. Taking into account that

$$(1+x)^{\alpha+\mu} R_k^{(\alpha+\mu,\alpha-\mu)}(x) = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 (1-y)^{\alpha} R_k^{(\alpha,\alpha)}(y)(y-x)^{\mu-1} dy$$

for $\alpha > -1$, $\mu > 0$, $x \in (-1, 1)$, we define

$$S_n^{(a,b)}(x) = \sum_{k=0}^{n} \alpha_k R_k^{(a,b)}(x), \quad \alpha_k \in \mathbb{R}, \quad k = 0,1,\ldots,n.$$

If $\alpha = \dfrac{a+b}{2}$ and $\mu = \dfrac{a-b}{2}$ then we obtain

COROLLARY 3. *If $x \in (-1, 1)$, $a > b \geq 0$ then the projection formulas*

$$S_n^{(a,b)}(x) = \frac{(1-x)^{-a}}{B\left(\frac{a-b}{2}, \frac{a+b}{2}+1\right)} \int_x^1 (1-y)^{\frac{a+b}{2}} S_n^{\left(\frac{a+b}{2},\frac{a+b}{2}\right)}(y)(y-x)^{\frac{a-b}{2}-1} dy$$

is valid.

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