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PROPERTIES OF WEIGHTED MEANS OF HIGHER ORDER

V. MIHEŞAN
(Cluj-Napoca)

Abstract. In [2] and [3], C. Mocanu studies the monotonicity, starshapedness and the convexity of Cesàro's weighted means of higher order.

We shall study the monotonicity, respectively the starshapedness with respect to a curve of these means, finding conditions in which Chebyshev's inequality holds for weighted means of higher order.

1. INTRODUCTION

A function $p:[0,a]\to R$, $a>0$ is called a *weight* on $[0,a]$ if the following conditions are satisfied:

1) p is continuous on $[0, a]$, $p(0)=0$,
2) p is continuous differentiable on $(0, a]$,
3) $p'(t)>0$ for every $t\in(0, a]$.

Let $L[0, a]$ be Lebesgue's space of the summable functions on $[0,a]$ and

$$L_p[0,a]=\{f|p'f\in L[0,a]\}.$$

For $f\in L_p[0,a]$ in [2] is defined the weighted mean

$$Af(x)=\frac{\int_0^x p'(t)\cdot f(t)\mathrm{d}t}{p(x)} \tag{1.1}$$

for $x\in(0,a]$ and $Af(0)=f(0)$.

The weighted mean of order n of f is defined inductively by

$$A_n f(x)=A(A_{n-1}f)(x),\ A_1=A, \tag{1.2}$$

where we suppose that $A_{n-1}f\in L_p[0,a]$.

2. THE MONOTONICITY AND THE STARSHAPEDNESS OF THE WEIGHTED MEANS OF HIGHER ORDER

PROPOSITION 2.1

$$(A_{n+1}f)(x) = \frac{p'(x)}{p(x)}\left(A_n f(x) - A_{n+1}f(x)\right). \tag{2.1}$$

Proof. The operator A, defined in (1.1) is inversable and has the inverse

$$A^{-1}f(x) = \frac{(pf)'(x)}{p'(x)}. \tag{2.2}$$

Then

$$A_n f(x) = A^{-1}A_{n+1}f(x) = \frac{(p\cdot A_{n+1}f)'(x)}{p'(x)} = $$

$$= A_{n+1}f(x) + \frac{p(x)}{p'(x)}(A_{n+1}f)'(x),$$

from which (2.1) is obtained.

COROLLARY 2.1. $A_{n+1}\uparrow \Leftrightarrow A_n f \geq A_{n+1}f$; $A_{n+1}\downarrow \Leftrightarrow A_n f \leq A_{n+1}f$.

Proof. From (2.1) it is obtained that

$$(A_{n+1}f)'(x) \geq 0 \Leftrightarrow A_n f \geq A_{n+1}f,$$

and the conclusion is obvious.

COROLLARY 2.2. *If $A_n\uparrow$, then $A_{n+1}\uparrow$. If $A_n\downarrow$, then $A_{n+1}\downarrow$*

Proof. If $A_n\uparrow$ from Corollary 2.1, it results that $A_{n-1}\geq A_n$. We apply A, which is monotone and we obtain $A_n \geq A_{n+1}$ and from Corollary 2.1 it follows that $A_{n+1}\uparrow$.

COROLLARY 2.3. *If $g>0$ then*

$$\frac{A_{n+1}f}{A_{n+1}g}\uparrow \Leftrightarrow \frac{A_{n+1}f}{A_{n+1}g} \leq \frac{A_n f}{A_n g}.$$

Proof. Let $h(x)=A_{n+1}f(x)/A_{n+1}g(x)$, which is increasing, hence

$$h'(x) = \frac{p'(x)}{p(x)}\cdot\frac{A_n f(x)\cdot A_{n+1}g(x) - A_{n+1}f(x)\cdot A_n g(x)}{A_{n+1}^2 g(x)} \geq 0,$$

from which the conclusion is obtained.

DEFINITION 2.1. *The function f is starshaped with respect to a curve (C) defined by $y=g(x)$ if the function $h(x)=f(x)/g(x)$ is increasing.*

Remark 2.1. For $g(x)=x$ the usual starshapedness is obtained.

PROPOSITION 2.2. *The weighted-mean of order $(n+1)$ is starshaped with respect to $p(x)$ if and only if $A_n f \geq 2A_{n+1}f$.*

Proof. $A_{n+1}f$ is starshaped with respect to $p(x)$ if the function $h(x)=A_{n+1}f(x)/p(x)$ is increasing, i.e. $h'(x) \geq 0 \Leftrightarrow (A_{n+1}f)'(x) \geq \frac{p'(x)}{p(x)}(A_{n+1}f)(x) \Leftrightarrow$

$$\Leftrightarrow \frac{p'(x)}{p(x)}\left[A_n f(x) - A_{n+1}f(x)\right] \geq \frac{p'(x)}{p(x)}A_{n+1}f(x) \Leftrightarrow A_n f(x) \geq 2A_{n+1}f(x).$$

3. CHEBYSHEV'S INEQUALITY FOR WEIGHTED MEANS OF HIGHER ORDER

Chebyshev's inequality

$$\int_0^x p'f\cdot\int_0^x p'g \leq p(x)\int_0^x p'fg \tag{3.1}$$

can be expressed with the operator defined in (1.1) by

$$Af(x)\cdot Ag(x) \leq Afg(x). \tag{3.2}$$

This inequality holds for different sets of functions: the set of the similarly ordered functions, the set of the functions whose averages are monotone [1]. We wish to give conditions in which Chebyshev's inequality holds for the means of $(n+1)$th order, i.e.

$$A_{n+1}f(x)\cdot A_{n+1}g(x) \leq A_{n+1}fg(x). \tag{3.3}$$

Let $H_n(x) = A_n fg(x) - A_n f(x)\cdot A_n g(x)$.

THEOREM 3.1. *If the inequalities:*

$$A\left[H_n(x)\right] \geq 0, \tag{3.4}$$

$$(A_n f - A_{n+1}f)(A_n g - A_{n+1}g) \geq 0, \tag{3.5}$$

hold, then (3.3) *is true.*

Proof. The derivative of $H_{n+1}(x)$ is

$$H'_{n+1}(x) = \frac{p'(x)}{p(x)}\left(A_n fg - A_{n+1} fg - A_{n+1}g\left(A_n f - A_{n+1} f\right) - A_{n+1} f\left(A_n g - A_{n+1} g\right)\right)$$

$$H'_{n+1}(x) = \frac{p'(x)}{p(x)}\left(H_n(x) - H_{n+1}(x) - \left(A_n f - A_{n+1} f\right)\left(A_n g - A_{n+1} g\right)\right)$$

We obtain

$$\frac{p(x)}{p'(x)} H'_{n+1}(x) - H_n(x) + H'_{n+1}(x) = \left(A_n f - A_{n+1} f\right)\left(A_n g - A_{n+1} g\right) \geq 0$$

$$p(x) \cdot H'_{n+1}(x) + p'(x) \cdot H_{n+1}(x) - p'(x) \cdot H_n(x) \geq 0$$

$$\frac{\left(p(x) \cdot H_{n+1}(x)\right)'}{p'(x)} \geq H_n(x).$$

From (3.2) it follows that

$$A^{-1} H_{n+1}(x) \geq H_n(x)\ ,\ H_{n+1}(x) \geq A\left[H_n(x)\right] \geq 0\ ,$$

hence $H_{n+1}(x) \geq 0$, which is (3.3).

Remark. For $n=0$, because $H_0(x)=0$, we obtain Theorem 1 from [1].

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Technical University
Department of Mathematics
3400 Cluj-Napoca, România