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ON THE METRIC PROJECTION AND THE QUOTIENT MAPPING

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1. INTRODUCTION

For a linear space Z and a nonvoid set U denote by Z^U the linear space (with respect to the pointwise operations of addition and multiplication by real scalars) of all applications from U to Z.

Let Y, X be two nonvoid sets such that $Y \subseteq X$ and let $\left(N_Y, \|\cdot\|_Y\right)$ and $\left(N_X, \|\cdot\|_X\right)$ be two normed spaces contained in Z^Y and Z^X respectively. Suppose that for every $F \in N_X$ the restriction $F|_Y$ of F to Y belongs to N_Y.

DEFINITON 1. We say that the norms $\|\cdot\|_Y$ and $\|\cdot\|_Y$ are compatible if

$$\left\| F|_Y \right\|_Y \leq \|F\|_X \tag{1}$$

for every $F \in N_X$.

In the following the norms $\|\cdot\|_Y$ and $\|\cdot\|_X$ will be supposed always compatible.

A nonvoid subset K of a normed space $(X, \|\cdot\|)$ is called a *cone* if:

a) $u+v \in K$, and
b) $\lambda \cdot u \in K$,

for all $u, v \in K$ and $\lambda \in R$, $\lambda \geq 0$.

DEFINITION 2. *Let K_Y and K_X be two cones in the linear spaces N_Y and N_X respectively. We say that the cone K_Y has the norm preserving extension ((NPE) in short) property with respect to K_X if $F|_Y \in K_Y$, for every $F \in K_X$ and every $f \in K_Y$ has a norm perserving extension $F \in K_X$* $\left(\text{i.e. } F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\right)$.

If K_Y has the (NPE)-property with respect to K_X let

(2) $$\mathscr{E}(f) := \{F \in K_X : F|_Y = f \text{ and } \|F\|_X = \|f\|_Y\}$$

denote the set of all (NPE) extensions of the function $f \in K_Y$.
Let also

(3) $$M_X := K_X - K_X$$

denote the liniar subspace of N_X generated by the cone K_X and let

(4) $$Y^{\perp} := \{G \in M_X : G|_Y = 0\}$$

be the annihilator of the set Y in the linear space M_X (remember that we have supposed $Y \subseteq X$).

Obviously that $Y^{\perp}$ is a closed linear sunspace of M_X.

A linear subspace V of normed space U is called *proximinal* if for every $u \in U$ *there exists* $v_0 \in V$ such that

$$\|u - v_0\| = d(u, V) := \inf\{\|u - v\| : v \in V\}.$$

The set

$$P_V(u) := \{v \in V : \|u - v\| = d(u, V)\}$$

is called the set of elements of best approximation for u by elements in V.

If $P_V(w) \neq \varnothing$ for all w in a subset W of U then the subspace V is called *W-proximinal*.

THEOREM 1. *If the cone K_Y has the (NPE)- property with respect to the cone K_X then:*

(a) The equality

(5) $$d(F, Y^{\perp}) = \|F|_Y\|_Y$$

is true for all $F \in K_X$;

(b) The inclusion

(6) $$F - \mathscr{E}(F|_Y) \subseteq P_{Y^{\perp}}(F)$$

holds for every $F \in K_X$;

(c) If furthermore $Y^{\perp} \subseteq K_X$, then

$$F - \mathscr{E}(F|_Y) = P_{Y^{\perp}}(F)$$

for all $F \in K_X$.

Proof. Let $F \in K_X$. For $G \in Y^{\perp}$ taking into account that the norms $\|\cdot\|_Y$ were supposed compatible in the sense of Definition 1, we have

$$\|F|_Y\|_Y = \|F|_Y - G|_Y\|_Y = \|(F - G)|_Y\|_Y \leq \|F - G\|_X .$$

It follows that

$$\|F|_Y\|_Y \leq d(F, Y^{\perp}).$$

On the other hand, because $F - H \in Y^{\perp}$ for $H \in \mathscr{E}(F|_Y)$, we have

$$\|F|_Y\|_Y = \|H\|_X = \|F - (F - H)\|_X \geq \inf\{\|F - G\|_X : G \in Y^{\perp}\} = d(F, Y^{\perp})$$

Combining the obtained inequalities one can write

$$d(F, Y^{\perp}) \leq \|F - (F - H)\|_X = \|F|_Y\|_Y \leq d(F, Y^{\perp})$$

It follows

$$\|F - (F - H)\|_X = d(F, Y^{\perp}) = \|F|_Y\|,$$

for every $H \in \mathscr{E}(F|_Y)$, proving formula (5) and inclusion (6).

In order to prove equality (7) suppose that $Y^{\perp} \subseteq K_X$ and let G be an arbitrary element of $P_{Y^{\perp}}(F)$. Since $G|_Y = 0$ and the norms $\|\cdot\|_Y$ and $\|\cdot\|_X$ are compatible, it follows that $(F - G)|_Y = F|_Y$ and, by (5),

$$\|F - G\|_X = d(F, Y^{\perp}) = \|F|_Y\|_Y ,$$

showing that F-G is a (NPE) extension of $F|_Y$. To prove that $F - G \in \mathscr{E}(F|_Y)$ it remains to show that $F - G \in K_X$ (see (2)). But $G \in Y^{\perp}$ implies $-G \in Y^{\perp}$ and, by hypothesis, $Y^{\perp} \subseteq K_X$ implying $-G \in K_X$, and, since K_X is a cone, $F-G=F+(-G) \in K_X$, showing that $F - G \in \mathscr{E}(F|_Y)$. This last relation is equivalent to $G \in F - \mathscr{E}(F|_Y)$ for every $G \in P_{Y^\perp}$ (F), .i.e. $P_{Y^\perp}(F) \subseteq F - \mathscr{E}(F|_Y)$, which together with inclusion (6), prove equality (7).

Taking $K_Y = N_Y$ and $K_X = N_X$ one obtains:

COROLLARY 1. *If N_Y has the (NPE)-propety with respect to N_X then*

(a') *The subspace* $Y^\perp = \{G \in N_X : G|_Y = 0\}$ *is proximal in* N_X *and*

(5') $$d(F, Y^\perp) = \|F|_Y\|_Y$$

for every $F \in N_X$.

(b') *The equality*

(7') $$P_{Y^\perp}(F) = F - \mathscr{E}(F|_Y)$$

holds for every $F \in N_X$.

Proof. It suffices to prove equality (7'). By Theorem 1 it follows that $F - \mathscr{E}(F|_Y) \subseteq P_{Y^\perp}(F)$. If $F \in N_X$ and $G \in P_{Y^\perp}(F)$ then

$$\|F - G\|_X = d(F, Y^\perp) = \|F|_Y\|_Y$$

Since $(F - G)_Y = F|_Y$ it follows that F-G is a norm preserving extension $F|_Y$. But $K_X = N_X$ implies $M_X = N_X$ so that F-$G \in M_X$ showing that $F - G \in \mathscr{E}(F|_Y)$ or equivalently $G \in F - \mathscr{E}(F|_Y)$ for every $G \in P_{Y_\perp}(F)$. This proves the inclusion $P_{Y_\perp}(F) \subseteq F - \mathscr{E}(F|_Y)$ and equality (7').

EXAMPLES

1°. Let $X = [a,b] \subset R$ and $Y = \{a,b\}$. Take $N_X = C[a,b]$ -the space of all realvalued continuous on $[a,b]$ with the sup-norm and $N_Y = C(\{a,b\})$.

Let K_X be the cone

$$K_X := \{F \in C[a,b] : F(a) = F(b) \geq 0\}$$

and $K_Y = K_X \cap C(\{a,b\})$, i.e.

$$K_Y := \{f \in C(\{a,b\}) : f(a) = f(b) \geq 0\}$$

Obviously that K_Y has the (NPE)-properly with respect to K_X. If $F \in K_X$ then $F|_Y \in K_Y$ and function $H(x) = f(a)$, $x \in [a,b]$, is an (NPE) extension of $f \in C(\{a,b\})$.

The space generated by th cone K_X is

$$M_X := K_X - K_X = \{F \in C[a,b] : F(a) = F(b)\}$$

and the annihilator space of the set Y in M_X is

$$Y^\perp := \{G \in M_X : G(a) = G(b) = 0\}.$$

By Theorem 1, the subspace $Y^\perp$ is K_X-proximal and $d(F, Y^\perp) = F(a)$ and $F - \mathscr{E}(F|_Y) \subseteq P_Y\perp(F)$, for each $F \in K_X$. We have

$$\mathscr{E}(F|_Y) = \{H \in K_X : H(a) = H(b) = \|H\|_X\} =$$
$$= \{H \in K_X : H(a) = H(b) \text{ and } |H(x)| \leq F(a), \text{ for all } x \in [a,b]\}.$$

It follows that $Y^\perp \subset K_X$ and therefore the equality $F - \mathscr{E}(F|_Y) = P_Y\perp(F)$ for each $F \in K_X$.

2°. Let $X = [-2,2] \subset R$, $Y = \{-1,0,1\}$ and let $N_X := \mathrm{Lip}_0[-2,2] = \{F : [-2,2] \to R : F$ is Lipschitz on $[-2,2]$ and $F(0) = 0\}$ equipped with the Lipschitz norm

$$\|F\|_X = \sup\{|F(x) - F(y)| / |x - y| : x, y \in [-2,2], x \neq y\}.$$

For N_Y take

$$N_Y := \mathrm{Lip}_0\{-1,0,1\} = \{f : \{-1,0,1\} \to R : f(0) = 0\}$$

equipped with the norm

$$\|f\|_Y = \max\{|f(-1)|, |f(1)|\}.$$

For K_X and K_Y take

$$K_X := \{F \in \mathrm{Lip}_0[-2,2] : F \text{ is convex on } [-2,2]\}$$
$$K_Y := \{f \in \mathrm{Lip}_0\{-1,0,1\} : f \text{ is convex on } \{-1,0,1\}\}.$$

By definition $F \in \mathrm{Lip}_0[-2,2]$ is in K_X if

$$F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y),$$

for every $x, y \in [-2, 2]$ and every $\lambda \in [0,1]$ and f is in K_Y if the divided difference $[-1,0,1; f]$ is nonnegative.

Obviously that K_Y has the (NPE)-property with respect to K_X: for $F\in K_X$ the restriction $F|_Y$ is in K_Y and $F(x)=\min_{y\in\{-1,0,1\}}\left[f(y)+\|f\|_Y|x-y|\right],\ x\in[-2,2]$ is a (NPE) extension in K_X of $f\in K_Y$.

Let

$$F(x)=\begin{cases}-x, & x\in[-2,0]\\ \frac{1}{2}x, & x\in(0,-2]\end{cases}$$

and

$$G(x)=\begin{cases}-x-1, & x\in[-2,-1)\\ 0, & x\in[-1,1]\\ \frac{1}{2}x-\frac{1}{2}, & x\in\left(1,\frac{1}{2}\right]\end{cases}$$

It follows that $G\in Y^{\perp}$ (in fact $G\in K_X\subseteq K_X-K_X$),

$$F(x)-G(x)=\begin{cases}1 & ,\ x\in[-2,-1]\\ -x & ,\ x\in(-1,0]\\ \frac{1}{2}x & ,\ x\in(0,1]\\ \frac{1}{2} & ,\ x\in(1,2]\end{cases}$$

and

$$\|F-G\|_X=\left\|F|_Y\right\|_Y=d\left(F,Y^{\perp}\right),\ (F-G)|_Y=F|_Y,$$

implying that F-G is a (NPE) extension of $F|_Y$. But F-G is not convex function on [-2, 2] showing that, in general, $F-\mathscr{E}\left(F|_Y\right)$ can be strictly contained in $P_{Y^{\perp}}(F)$. Therefore equality (7) is not true without any supplementary hypotheses on the space $Y^{\perp}$.

3°. Let X, Y, N_X, N_Y be as in Example 2° and

$$Y^{\perp}=\left\{G\in N_X:G|_Y=0\right\}$$

By Mc Shane's theorem (see [9]), the space N_Y has the (NPE)-property with respect to N_X. By Corollary the subspace $Y^{\perp}$ is proximinal in N_X and every element G of best approximation of a function $F\in N_X$ by elements in $Y^{\perp}$ has the form $G=F-H$ for a function $H\in \mathscr{E}\left(F|_Y\right)$. Also $d\left(F,Y^{\perp}\right)=\left\|F|_Y\right\|_Y$ and $P_{Y^{\perp}}(F)=F-\mathscr{E}\left(F|_Y\right)$.

Other examples to which Corollary 1 applies are given by Hahn-Banach extension theorem, by Tietze extension theorem, by Helly extension Theorem (see [7], [8], [12], [16]).

2. THE QUOTIENT MAPPING

Consider the quotient subspace $M_X/Y^{\perp}$ with respect to its subspace $Y^{\perp}$, defined by

(8) $$M_X/Y^{\perp}=\left\{F+Y^{\perp}:F\in M_X\right\}.$$

Since the subspace $Y^{\perp}$ is closed in M_X it follows that

(9) $$\left\|F+Y^{\perp}\right\|=d\left(F,Y^{\perp}\right),\ F\in M_X,$$

is a norm on $M_X/Y^{\perp}$.

Let

(10) $$K_X/Y^{\perp}:=\left\{F+Y^{\perp}:F\in K_X\right\}$$

and let

(11) $$\operatorname{Ker}P_{Y^{\perp}}|_{K_X}:=\left\{F\in K_X:0\in P_{Y^{\perp}}(F)\right\}$$

be the kernel of the restriction of the metric projection $P_{Y^{\perp}}$ to K_X.

Obviously that

$$\operatorname{Ker}P_{Y^{\perp}}|_{K_X}=\left\{F\in K_X:\|F\|_X=d\left(F,Y^{\perp}\right)\right\}=\left\{F\in K_X:\|F\|_X=\left\|F|_Y\right\|_Y\right\}.$$

The application

(12) $$Q_{K_X}:K_X\to K_X/Y^{\perp},\ Q(f)=F+Y^{\perp},\ F\in K_X,$$

is called the *quotient mapping* of the cone K_X onto the cone $K_X/Y^{\perp}$.

THEOREM 2. *If the cone K_Y has the (NPE)-property with respect to the cone K_X then*

$$1^{\circ}.\ K_X\subseteq Y^{\perp}+\operatorname{Ker}P_{Y^{\perp}}|_{K_X}=\left\{G+H:G\in Y^{\perp},G\in\operatorname{Ker}P_{Y^{\perp}}|_{K_X}\right\};$$

$2^{\circ}.\ Q\left(\mathrm{Ker}P_{Y^{\perp}}\big|_{K_X}\right) = K_X / Y^{\perp}$;

$3^{\circ}.\ F - \left(Q\big|_{\mathrm{Ker}P_{Y^{\perp}}|_{K_X}}\right)^{-1}\left(F + Y^{\perp}\right) \subseteq P_{Y^{\perp}}\big|_{K_X}(F)$, for every $F \in K_X$.

Proof. **1°.** By Theorem 1. (b), each $F \in K_X$ there has an element of best approximation $G \in Y^{\perp}$ and $G = F - H$ for a function $H \in \mathscr{E}\left(F|_Y\right)$. It follows that $\|H\|_X = \left\|F|_Y\right\|_Y = d\left(H, Y^{\perp}\right)$ implying $H \in \mathrm{Ker}P_{Y^{\perp}}\big|_{K_X}$; and then $F = G + Y^{\perp}$, $H \in \mathrm{Ker}P_{Y^{\perp}}\big|_{K_X}$.

2°. Let $F + Y^{\perp} \in K_X / Y^{\perp}$. Since $F|_Y \in K_Y$ and K_Y has the (NPE)-property with respect to K_X it follows that for every $H \in \mathscr{E}\left(F|_Y\right) \subseteq \mathrm{Ker}P_{Y^{\perp}}\big|_{K_X}$ we have

$$Q(H) = H + Y^{\perp} = F + Y^{\perp},$$

because $F - H \in Y^{\perp}$. Therefore $Q_{\mathrm{Ker}P_{Y^{\perp}}|_{K_X}}$ is a surjection.

3°. By 2°, the application

$$E = \left(Q_{\mathrm{Ker}P_{Y^{\perp}}|_{K_X}}\right)^{-1} : K_X / Y^{\perp} \to 2^{\mathrm{Ker}P_{Y^{\perp}}|_{K_X}}, \tag{13}$$

given by

$$E\left(F + Y^{\perp}\right) = \mathscr{E}\left(F|_Y\right) \subset \mathrm{Ker}P_{Y^{\perp}}\big|_{K_X}, \tag{14}$$

is well defined.

But then, by Theorem 1. (b), it follows that

$$F - \mathscr{E}\left(F|_Y\right) \subseteq P_{Y^{\perp}}\big|_{K_X}(F).$$

Theorem 2 is proved.

Using Theorem 2 one can obtain some relations between the properties of the selections associated to the metric projection $P_{Y^{\perp}}\big|_{K_X}$ and the properties of the selections associated to the application E defined by (13) and (14). (see [3], [12], [13], [16]).

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