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ON COMPUTATIONAL COMPLEXITY IN SOLVING EQUATIONS BY STEFFENSEN-TYPE METHODS

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1. INTRODUCTION

This note is a continuation of the paper [4]. We shall establish here the optimal methods for the efficiency index of the class of Steffensen-type methods.

We adopt the efficiency index of an iterative process as being the number $I(\omega, p)$ given in [1] by:

$$I(\omega, p) = \omega^{\frac{1}{p}} \tag{1.1}$$

where ω is the convergence order of the iterative method and p represents the number of function evaluations that must be performed at each step. As it results from [1] and [4], the efficiency index can be defined as in (1.1) if we admit that the number of function evaluations is constant beginning from a certain step.

Let $I \subset R$ denote an interval of the real axis, and consider the equation

$$f(x) = 0, \tag{1.2}$$

where $f: I \to R$. Suppose that equation (1.2) possesses a unique root $\bar{x} \in I$. Also suppose that f admits derivatives up to the order $m+1$, $m \in \mathbb{N}$, the $(m+1)$-th derivative of f is bounded on I, and $f'(x) \neq 0$ for all $x \in I$. If $F = f(I)$, then there exists the function $f^{-1}: F \to I$ and $\bar{x} = f^{-1}(0)$.

It is obvious that for approximating the solution of (1.2) it is sufficient to approximate f^{-1} at $y = 0$.

From the derivability hypotheses concerning f it follows that f^{-1} also possesses derivatives up to the order $m+1$, which are given by [2]:

$$\left[f^{-1}(y)\right]^{(k)} = \sum \frac{(2k-2-i_1)!(-1)^{k-1+i_1}}{i_2!\,i_3!\ldots i_k!\left[f'(x)\right]^{2k-1}}\left(\frac{f'(x)}{1!}\right)^{i_1}\cdots\left(\frac{f^{(k)}(x)}{k!}\right)^{i_k} \tag{1.3}$$

$k = \overline{1, m+1}$, where the above sum extends over all the integer nonnegative solutions of the system:

$$i_2 + 2i_3 + \dots + (k-1)i_k = k - 1, \quad (1.4)$$
$$i_1 + i_2 + \dots + i_k = k - 1.$$

We shall consider the following general iterative process for solving the equation (1.2):

$$x_{n+k+1} = g(x_k, x_{k+1}, \dots, x_{k+n}), \quad n \geq 0,\ k = 1,2,\dots, \quad (1.5)$$

where $g : I^{n+1} \to I$ is a function whose restriction to the diagonal of I^{n+1} coincides with a function $h: I \to I$, whose fixed point is $\bar{x}$, i.e. $g(x, x, \dots, x) = h(x)$ for all $x \in I$ and $h(\bar{x}) = \bar{x}$.

In order to establish the optimal efficiency index of the class of Steffensen methods we shall adopt, as in [4], the following assumptions:

We consider as a function evaluation :

a) the evaluation of the function or of any of its derivatives at a certain point;

b) the evaluation by (1.3) of any of the derivatives of f^{-1} at a certain point;

c) the evaluation of g from (1.5) at a certain point.

2. GENERALIZED STEFFENSEN METHOD

Let :

$$x_1, x_2, \dots, x_{n+1} \quad (2.1)$$

be n+1 interpolation nodes from I and

$$y_1, y_2, \dots, y_{n+1} \quad (2.2)$$

the values of f at x_i, $y_i = f(x_i)$, $i = \overline{1, n+1}$.

Consider n+1 natural numbers $a_1, a_2, \dots, a_{n+1}$ such that $a_i \geq 1$ $i = \overline{1, n+1}$, and $a_1 + a_2 + \dots + a_n + a_{n+1} = m+1$. Supposing that at each x_i, $i = \overline{1, n+1}$, we know the values of f and of its derivatives up to the order a_i-1, i.e. we know $f(x_i), f'(x_i), \dots, f^{(a_i-1)}(x_i)$, by (1.3) we can get the values of f^{-1} and of its derivatives up to the order $a_i - 1$.

We can now construct the Hermite inverse interpolation polynomial corresponding to f^{-1}, nodes (2.2), i.e. the following polynomial exists and is unique:

$$H\left(y_1, a_1; y_2, a_2; \dots; y_{n+1}, a_{n+1}; f^{-1} \middle| y\right) = \quad (2.3)$$
$$= \sum_{i=1}^{n+1} \sum_{j=0}^{a_i-1} \sum_{k=}^{a_i-j-1} \left[f^{-1}(y_i)\right]^{(j)} \frac{1}{k!\,j!} \left[\frac{(y-y_i)^{a_i}}{\omega(y)}\right]^{(k)}_{y=y_i} \frac{\omega(y)}{(y-y_i)^{a_i-j-k}}$$

where

$$\omega(y) = (y-y_1)^{a_1}(y-y_2)^{a_2}\cdots(y-y_{n+1})^{a_{n+1}}. \quad (2.4)$$

If x_{n+2} denotes the value of H at $y = 0$ we have

$$|\bar{x} - x_{n+2}| \leq \frac{M_{m+1}}{(m+1)!} |f(x_1)|^{a_1} |f(x_2)|^{a_2} \cdots |f(x_{n+1})|^{a_{n+1}}, \quad (2.5)$$

where $M_{m+1} = \sup_{y \in F} \left| \left[f^{-1}(y)\right]^{(m+1)} \right|$.

If $x_k, x_{k+1}, \dots, x_{k+n} \in I$ are n+1 approximations of $\bar{x}$, then a new approximation x_{k+n+1} can be obtained by (2.3):

$$x_{k+n+1} = H(y_k, a_1;\ y_{k+1}, a_2;\ \dots;\ y_{k+n}, a_n;\ f^{-1} \mid 0),\ k = 1, 2, \dots, \quad (2.6)$$

with the error evaluation

$$|\bar{x} - x_{k+n+1}| \leq \frac{M_{m+1}}{(m+1)!} |f(x_k)|^{a_1} |f(x_{k+1})|^{a_2} \cdots |f(x_{k+n})|^{a_{n+1}}. \quad (2.7)$$

Method (2.6) is called Hermite-like iterative method.

Consider a function $\varphi : I \to I$ whose fixed point from I is $\bar{x}$ i.e. $\varphi(\bar{x}) = \bar{x}$, and suppose there exists a real number $\alpha > 0$ such that

$$|f(\varphi(x))| \leq \alpha |f(x)|, \ \text{ for all } \ x \in I. \quad (2.8)$$

Let $\varphi_1(x) = \varphi(x)$, $\varphi_2(x) = \varphi(\varphi_1(x))$, $\varphi_3(x) = \varphi(\varphi_2(x))$, ..., $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$, be the iterations up to the order n of the function φ.

To increase the convergence order of method (2.6) we can do as it follows. Let $x_k \in I$ be a certain approximation of the solution $\bar{x}$ of equation (1.2) and $u_k = x_k$, $u_{k+1} = \varphi_1(x_k)$, ..., $u_{n+k} = \varphi_n(x_k)$. Consider the values $\bar{y}_i = f(u_i)$ $i = \overline{k, n+k}$ as interpolation nodes in (2.3). Then x_{k+1}, the next approximation of $\bar{x}$, is given by:

$$x_{k+1} = H\left(\bar{y}_k, a_1; \bar{y}_{k+1}, a_2; \dots; \bar{y}_{k+n}, a_{n+1}; f^{-1} \middle| 0\right). \quad (2.9)$$

Repeating this process, called Steffesen type iterative method, we obtain a sequence $(x_n)_{n \geq 0}$ of approximations for $\bar{x}$.

Using (2.8) and (2.7) it can be easily seen that the convergence order of (2.9) is $m+1$.

3. THE EFFICIENCY INDEX OF STEFFENSEN-TYPE METHODS

As it can be seen above, at each iteration step in (2.9) we have the following function evaluations:

1) n values of φ to obtain the interpolation nodes u_{k+i}, $i=\overline{1,n}$;

2) $n+1$ values of f at the nodes u_{k+i} , $i=\overline{0,n}$;

3) at each interpolation node u_{k+i} , $i=\overline{0,n}$ we compute the values of successive derivatives of f up to the order $a_{i+1}-1$, altogether $m-n$ function evaluations;

4) by (1.3) we evaluate the successive derivatives of f^{-1} at $\overline{y}_{k+1}=f(u_{k+i})$, $i=\overline{0,n}$ up to the order $a_{i+1}-1$, altogether $m-n$ function evaluations;

5) finally, consider (2.9) as a single function evaluation.

Summing up, we obtain altogether $2(m+1)$ function evaluations.

Using (1.1) we obtain the following expression for the efficiency index of the class of Steffensen-type methods:

$$I(m+1,\,2(m+1))=(m+1)^{\frac{1}{2(m+1)}} \tag{3.1}$$

Elementary considerations on the behaviour of the function $h:(0,+\infty)\to R$, $h(t)=t^{\frac{1}{2t}}$ lead us to the conclusion that the function $I(m+1,\,2(m+1))$ attains its maximum at $m=2$.

Note that the efficiency index (3.1) does not depend on the number of interpolation nodes.

From $m=2$ and $a_1+a_2+\ldots+a_{n+1}=m+1$, $a_i\geq 1$ $i=\overline{1,n+1}$ it follows that $n\leq 2$.

We shall successively analyse all the cases that lead us to the optimal methods from (2.9).

A. $a_1+a_2+a_3=3$, i.e. $a_1=a_2=a_3=1$. Then (2.3) becomes the Lagrange's inverse interpolation polynomial, and (2.9) is written:

$$x_{k+1}=x_k-\frac{f(x_k)}{[x_k,\varphi(x_k);f]}-\frac{[x_k,\varphi(x_k),\varphi(\varphi(x_k));f]\,f(x_k)\,f(\varphi(x_k))}{[x_k,\varphi(x_k);f][x_k,\varphi(\varphi(x_k));f][\varphi(x_k),\varphi(\varphi(x_k));f]}$$

$$x_0\in I,\ k=0,1,\ldots, \tag{3.2}$$

where $[u,v;f]$ respectively $[u,v,w;f]$ denote the first, respectively the second order divided differences of f.

B. $a_1+a_2=3$, i.e. $a_1=2$, $a_2=1$ or $a_1=1$ and $a_2=2$. When $a_1=2$, $a_2=1$ we obtain the following method:

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}+\frac{\left(f'(x_k)-[x_k,\varphi(x_k);f]\right)f^2(x_k)}{[x_k,\varphi(x_k);f]^2\cdot f'(x_k)(\varphi(x_k)-x_k)}, \tag{3.3}$$

$$x_0\in I,\ k=0,1,\ldots,$$

and when $a_1=1$, $a_2=2$ it follows:

$$x_{k+1}=\varphi(x_k)-\frac{f(\varphi(x_k))}{f'(\varphi(x_k))}+\frac{\left([x_k,\varphi(x_k);f]-f'(\varphi(x_k))\right)f^2(\varphi(x_k))}{[x_k,\varphi(x_k);f]^2\,f'(\varphi(x_k))(\varphi(x_k)-x_k)}, \tag{3.4}$$

$$x_0\in I,\ k=0,1,\ldots,$$

C. $a_1=3$. In this case we get from (2.9) the third order Chebyshev iterative method, studied in [4].

In conclusion, the following theorem holds:

THEOREM 3.1. *Under the assumptions a) – c) from 1., in the class of Steffensen-type iterative methods any of the methods (3.2), (3.3) or (3.4) is optimal, i.e. has the greatest efficiency index.*

Remark. For the particular case when $a_1=a_2=\ldots=a_{n+1}=q$ the condition of optimality for the efficiency index gives us two possibilities, namely $q=3$, $n=0$, hence the case **C.** *or $q=1$, $n=2$, hence the case* **A.**

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