REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 24, N^os 1–2, 1995, pp. 221–224

CORRECTION IN THE DOMINANT SPACE METHOD FOR THE HEAT EQUATION

TITUS PETRILA and DAMIAN TRIF
(Cluj-Napoca)

1. INTRODUCTION

Let us consider an equation of the type

$$u_t = Lu + f \text{ on } \Omega \times (0, T), \Omega \subset R^d$$

with appropriate initial and boundary conditions. Here we denoted by L a differential operator in space while f is a given function. The unknown function u is assumed to be sufficiently smooth.

By spatial discretization, using spectral methods with N_i, $i = \overline{1, d}$ points in each dimension, we get a set of ordinary differential equations

$$V_t = F(t, V), V(t) \in R^N, F: R \times R^N \to R^N$$

with $N = N_1 \text{x} \ldots \text{x} N_d$. This system is large and stiff, which implies serious difficulties both in storage and effective computation of the solutions.

Taking into account the fact that the stiffness character as well as an explicit approach lead to severe restriction for the time-step, the implicit methods seem to be the best fit to such problems. Our goal is to improve the mentioned implicit procedure by considering some "almost" explicit schemes. Precisely a pure implicit procedure will be considered only for the dominant directions while the rest of the system is explicitly solved. We also remark that the limitation of the time-step will be observed from the point of view of a better accuracy and not any more of the stability condition.

2. CORRECTION IN THE DOMINANT SPACE METHOD

We present this method on the $2D$ heat equation

$$u_t = u_{xx} + u_{yy} \quad \text{on } \Omega = (-1, 1) \times (-1, 1), \ t > 0$$
$$u|_{\partial\Omega} = 0$$

$$u(x, y, 0) = u_0(x, y)$$

Using for the spatial discretization the Chebyshev-collocation spectral method, on the Gauss-Lobatto points $x_i = \cos(i\pi/N)$, $y_j = \cos(j\pi/N)$, $i, j = 0,1,\ldots,N$, we get

$$(1)\qquad \begin{aligned} U' &= D_0^2 U + U D_0^{2T} \\ U(0) &= U_0 \end{aligned}$$

where U is the matrix $u(x_i, y_j, t)$, $i, j = 1,\ldots, N-1$, D_0^2 is the second order collocation derivation matrix on the Gauss-Lobatto points with Dirichlet conditions and D_0^{2T} is the transpose of D_0^2. One has, cf.[1], for $i = 0,1,\ldots,N$, $j = 0,1,\ldots,N$

$$(D_N)_{i,j} = \begin{cases} \dfrac{c_i}{c_j}\dfrac{(-1)^{i+j}}{x_i - x_j} & i \neq j \\ \dfrac{-x_j}{2(1-x_j^2)} & 1 \le i = j \le N-1 \\ \dfrac{2N^2+1}{6} & i = j = 0 \\ -\dfrac{2N^2+1}{6} & i = j = N \end{cases}$$

and $D_0^2 = \left(\left(D_N^2\right)_{i,j}, i, j = 1,\ldots,N-1\right)$, where

$$c_i = \begin{cases} 2 & i = 0, N \\ 1 & i = 1,\ldots,N-1 \end{cases}$$

The exact solution of system (1), in the matrix form, is

$$(2)\qquad U(t_n + h) = e^{D_0^2 h}\, U(t_n)\, e^{D_0^{2T} h}$$

for $n = 0, 1, \ldots$ where $t_0 = 0$ and $t_{n+1} = t_n + h$ [2]

Let λ_k, v_k, w_k, $k = 1, N-1$ be the eigenvalues, the right and the left (column) eigenvectors of D_0^2, i.e.

$$D_0^2 v_k = \lambda_k v_k\,,\quad w_k^T D_0^2 = \lambda_k w_k^T\,,\quad k = 1,\ldots,N-1$$

It is shown [3] that λ_k are distinct, negative real numbers and consequently

$$(3)\qquad e^{D_0^2 h} = \sum_{k=1}^{N-1} e^{\lambda_k h}\, v_k w_k^T$$

For large N, the computation and storage of whole basis of the eigenvectors and eigenvalues as well as of $\exp(D_0^2 h)$ could be difficult.

The explicit numerical methods for system (1) essentially replace matrix (3) with a partial sum

$$\sum_{k=0}^{M} \frac{\left(D_0^2\right)^k h^k}{k!}$$

Since some of the eigenvalues are of order $O(N^4)$, it is necessary to choose a very small time-step h in order to ensure both the stability and the accuracy . Through the correction in the dominant space method [4], the solution of system (1) is approximated by some explicit method followed by the corrections on the dominant direction. By this correction, the coefficients of the explicit solution corresponding to large (in modulus) eigenvalues λ_k are replaced by the coefficients of the exact solution or by the coefficients arising from an unconditionally stable implicit method.

In the case of the modified Euler method, the algorithm is

$$(4)\qquad \begin{aligned} T &= I + D_0^2 + \frac{\left(D_0^2\right)^2 h^2}{2} + \sum_{k=1}^{M}\left(e^{\lambda_k h} - 1 - \lambda_k h - \frac{\lambda_k^2 h^2}{2}\right) v_k w_k^T \\ U^{n+1} &= T\, U^n\, T^T, \quad n = 0, 1, \ldots \end{aligned}$$

where $\lambda_1, \ldots, \lambda_M$ are the dominant eigenvalues (i.e. real, negative and maximal in modulus).

The matrix T is computed in a preprocessing stage and afterwards the procedure becomes explicit. The restrictions that stability imposes on the time-step are just those that would arise if the dominant eigenvalues were not present.

3. NUMERICAL EXAMPLES

As an example, in the case of $u_0(x,y) = \sin\frac{\pi}{2}(x+1)\sin\pi(y+1)$, with the exact solution $u(x,y,t) = \exp(-5\pi^2 t/4)\sin\frac{\pi}{2}(x+1)\sin\pi(y+1)$ we get the following data

N	M	$\lambda_1,\dots\lambda_M$	with correction		without correction	
			h	max. err.	h	max. err.
8	2	-214.3723 -201.6042	0.025	0.002	0.01	0.004
16	2	-3174.79 -3137.74	0.003	0.00003	0.0006	$<10^{-6}$
24	4	-15869.8 -15790.7 -2831.7 -2821.2	0.0016	$<10^{-6}$	0.000125	$<10^{-6}$

where h is the biggest time-step chosen to ensure the stability.

In the presence of a source term or nonhomogeneous Dirichlet conditions, the discretized system (1) becomes nonhomogeneous

$$U' = D_0^2 U + U D_0^{2T} + F$$

$$U(0) = U_0$$

with the exact solution

$$U(t_n + h) = e^{D_0^2 h} U(t_n) e^{D_0^2 T_h} + \int_{t_n}^{t_n+h} e^{D_0^2(t_n+h-s)} F(s) e^{D_0^2 T(t_n+h-s)} ds$$

If the trapezoidal rule is used, an approximation of the solution is

$$u(t_n + h) = e^{D_0^2 h}\left(u(t_n) + \frac{h}{2} F(t_n)\right) e^{D_0^{2T} h} + \frac{h}{2} F(t_n + h)$$

and algorithm (4) becomes

$$U^{n+1} = T\left(U^n + \frac{h}{2} F(t_n)\right) T^T + \frac{h}{2} F(t_{n+1})$$

with the same T. We note that the time-step can be changed without any matrix inversion matters, it being constrained only to conditions arising from accuracy.

REFERENCES

1. Canuto, C. et al., *Spectral Methods in Fluid Dynamics*, Springer, 1988.
2. Bellman, R., *Introduction to Matrix Analysis*, McGraw-Hill, 1960.
3. Gottlieb, D., Lustman, L., *The spectrum of the Chebyshev collection operator for the heat equation*, SIAM J. Numer. Anal., **20**, *5* (1983), pp. 909-921.
4. Alfred, P., Lambert, J.D., *Correction in the Dominant Space: A Numerical Technique for a Certain Class of Stiff Initial Value Problems*, Math. Comp., **31**, *140* (1977), pp. 922-938.

Received 24 XI 1994

Faculty of Mathematics,
University of Cluj-Napoca,
3400 Cluj-Napoca
România