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ON A PROCESS FOR OBTAINING ITERATIVE FORMULAS OF HIGHER ORDER FOR ROOTS OF EQUATIONS

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1. Let

(1) $$x_{n+1}=f(x_n),\ n=0,1,2,\ldots$$

be an iterative method for finding the root $x=a$ of the real or complex equation $F(x)=0$.

For the iterative method (1) which converges to $x=a$, we say it is of order k if

(2) $$|x_{n+1}-a| = O(|x_n-a|^k),\ n\to\infty .$$

If the function $f(x)$ is k times differentiable in a neighborhood of the limit point $x=a$, then the iterative method (1) is of order k if and only if

(3) $$f(a)=a, f'(a)=f''(a)=\ldots=f^{(k-1)}(a)=0,\ f^{(k)}(a)\neq 0.$$

2. In [6] Theorem 1 is given which represents the consequence of a theorem proved in [1]. In [2] Theorem 2 is proved. We state here these theorems.

THEOREM 1. *Let* (1) *be an iterative method of order* k (≥ 2), *and let the function* $f(x)$ *be* $k+1$ *times differentiable in a neighborhood of the limit point* $x=a$. *Then*

$$x_{n+1}=f(x_n)-\frac{1}{k}f'(x_n)(x_n-f(x_n))=$$

(4) $$=x_n-\left(1+\frac{1}{k}f'(x_n)\right)(x_n-f(x_n)),\ n=0,1,2,\ldots$$

is an iterative method of order at least $k+1$.

THEOREM 2. *Let* (1) *be an iterative method of order* k. *Let the function* $f(x)$ *be* $k+1$ *times differentiable in a neighborhood of the limit point* $x=a$ *and let* $f'(a)\neq k$. *Then*

$$(5)\qquad x_{n+1} = \frac{f(x_n) - \frac{1}{k} f'(x_n) x_n}{1-\frac{1}{k} f'(x_n)} = x_n - \frac{x_n - f(x_n)}{1-\frac{1}{k}f'(x_n)}, \quad n=0,1,2,\dots$$

is an iterative method of order at least $k+1$.

3. In this paper we also give an iterative process by which, starting from an iterative method of order k, one obtains an iterative method of order at least $k+1$. In this connection the following theorem is proved here.

THEOREM 3. *Let* (1) *be an iterative method of order* k. *Let the function* $f(x)$ *be* $k+1$ *times differentiable in a neighbourhood of the limit point* $x = a$ *and let* $f'(a) \neq 1$. *Then*

$$(6)\qquad x_{n+1} = f(x_n) - \frac{1}{k} f'(x_n)\left(\frac{x_n - f(x_n)}{1-\frac{1}{k}f'(x_n)}\right),$$

that is

$$(7)\qquad x_{n+1} = x_n - \left(1+\frac{1}{k}\left(\frac{f'(x_n)}{1-f'(x_n)}\right)\right)\left(x_n - f(x_n)\right), \quad n=0,1,2,\dots$$

is an iterative method of order at least $k+1$.

Proof of Theorem 3. In the method (1) the iterative function is $f(x)$, and in the method (6) the iterative function is

$$(8)\qquad g(x) = f(x) - \frac{1}{k} f'(x)\left(\frac{x-f(x)}{1-f'(x)}\right).$$

For the function $g(x)$ we shall prove that

$$(9)\qquad g(a) = a,\ g'(a) = g''(a) = \dots = g^{(k)}(a) = 0.$$

By hypothesis, (1) is an iterative method of order k and therefore relations (3) hold.

In view of, we obtain from (8)

$$(10)\qquad g(a) = a.$$

From (8) we have

$$g^{(r)}(x) = f^{(r)}(x) - \frac{1}{k}\left[f^{(r+1)}(x)\left(\frac{x-f(x)}{1-f'(x)}\right) + \binom{r}{1} f^{(r)}(x)\left(\frac{x-f(x)}{1-f'(x)}\right)' + \right.$$

$$(11)\qquad \left. + \binom{r}{2} f^{(r-1)}(x)\left(\frac{x-f(x)}{1-f'(x)}\right)'' + f'(x)\left(\frac{x-f(x)}{1-f'(x)}\right)^{(r)}\right].$$

In view of relations (3), we obtain from (11)

$$(12)\qquad g^{(r)}(a) = 0 \quad \text{for } r = 1, 2, \dots, k-1.$$

Since

$$\left(\frac{x-f(x)}{1-f'(x)}\right)' = \frac{\left(1-f'(x)\right)^2 + \left(x-f(x)\right)f''(x)}{\left(1-f'(x)\right)^2}$$

and keeping in mind relations (3), from(11) for $r = k$ we obtain

$$(13)\qquad g^{(k)}(a) = f^{(k)}(a) - \frac{1}{k}\left[k f^{(k)}(a)\right] = 0.$$

From (10), (12) and (13) we conclude that conditions (9) are fulfilled, which means that the iterative method (6) is of order at least $k+1$, which ends the proof of Theorem 3.

For $k = 1$ the iterative methods (5) and (6) coincide and reduce to

$$x_{n+1} = x_n - \frac{x_n - f(x_n)}{1-f'(x_n)}, \quad n = 0,1,2,\dots$$

4. *An example.* If (1) represents Newton's method for finding simple roots of the equation $F(x) = 0$, namely

$$(14)\qquad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}, \quad n = 0,1,2,\dots$$

which means that

$$f(x_n) = x_n - \frac{F(x_n)}{F'(x_n)},$$

then from (4), (5), (6) we obtain the following methods, respectively:

$$(15)\qquad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2\left(F'(x_n)\right)^2 + F(x_n)F''(x_n)}{2\left(F'(x_n)\right)^2},$$

$$(16)\qquad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2\left(F'(x_n)\right)^2}{2\left(F'(x_n)\right)^2 - F(x_n)F''(x_n)},$$

$$(17)\qquad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2\left(F'(x_n)\right)^2 - F(x_n)F''(x_n)}{2\left(F'(x_n)\right)^2 - 2F(x_n)F''(x_n)},$$

$(n = 0,1,2,\dots)$.

According to the preceding theorems, the iterative methods (15), (16), (17) are of order 3, since as we know Newton's method, (14) is of order 2.

Method (15) is known as Chebyshev's iterative method (see [3]). The asymptotic error constant for the iterative method (15) is

$$C_3 = \frac{3(F''(a))^2 - F'(a)F'''(a)}{6(F'(a))^2}.$$

Method (16) represents Halley's iterative method (see [4] and [5]). The asymptotic error constant for the iterative method (16) is

$$C_3 = \frac{3(F''(a))^2 - 2F'(a)F'''(a)}{12(F'(a))^2}.$$

For the iterative method (17) the asymptotic error constant is

$$C_3 = \frac{F'''(a)}{6(F'(a))^2}.$$

5. Methods (15), (16), (17) are special cases of family iterative methods

$$(18) \quad x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)} \cdot \frac{2(F'(x_n))^2 - sF(x_n)F''(x_n)}{2(F'(x_n))^2 - (s+1)F(x_n)F''(x_n)}, \quad n = 0,1,2,\ldots$$

where s is a finite parameter.

For simple roots of the equation $F(x) = 0$ the order of the iterative method (18) is 3 for every fixed finite value of the parameter s, which is easily verified. The asymtotic error constant for the iterative method (18) is

$$C_3 = \frac{3(1-s)(F''(a))^2 - 2F'(a)F'''(a)}{12(F'(a))^2}.$$

For $s = -1$ from (18) we obtain Chebyshev's method (15), and for $s = 0$ from (18) we obtain Halley's method (16). For $s = 1$ from (18) we obtain method (17).

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