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ON THE BETA APPROXIMATING OPERATORS OF SECOND KIND

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1. INTRODUCTION

In this paper we continue our earlier investigations [12], [13], [16] concerning the use of probabilistic methods for constructing linear positive operators useful in approximation theory of functions. By starting from the beta distribution of second kind $b_{p,q}$ (with positive parameters), which belongs to Karl Pearson's Type VI, one defines at (3) the beta second-kind transform $T_{p,q}$ of a function $g : [0, \infty) \to \mathbb{R}$, bounded and Lebesgue measurable in every interval $[a,b]$, where $0 < a < b < \infty$, such that $T_{p,q} \ |g| < \infty$. At (4) is given an explicit expression for the moment of order r $(1 \leq r < q)$ of the functional $T_{p,q}$. If one applies this transform to the image of a function $f : [0, \infty) \to \mathbb{R}$, by the Baskakov operator $\mathscr{B}_m$, defined at (8), we obtain the functional $F_m(p,q) = T_{p,q}(\mathscr{B}_m f)$, given explicitly at (9). If we choose $p = x/\alpha$, $q = 1/\alpha$, where α is a positive parameter, then we arrive at a parameter-dependent operator $L_m^{(\alpha)}$, introduced in 1970 in our paper [14] (see also [15]), as a generalization of the Baskakov operator.

The main result of this paper consists in introducing and investigating the approximation properties of a new beta operator of a second kind $L_m = T_{mx,\, m+1}$, which is an integral linear positive operator reproducing the linear functions.

As we have mentioned in the final part of the paper, this operator is distinct from the other beta type operators used so far in approximation theory of functions.

At (13), (14) and (15) we gave estimations of the orders of approximation, by using the moduli of continuity of first and second orders. At (16) we gave an asymptotic formula of Voronovskaja type, while at (18) and (19) we established two representations for the remainder term of the approximation formula (17).

2. SOME RESULTS ON THE PROBABILISTIC METHODS USED FOR CONSTRUCTION OF LINEAR POSITIVE OPERATORS

One denotes by $F_{m,x}$ a family of probability distribution functions having the expectation $x \in \mathbb{R}$ and the variance $\delta_m^2(x)$, for any $m \in \mathbb{N}$. Let $Z_{m,x}$ be a random variable with the distribution function $F_{m,x}$ and f a single-valued function $f: \mathbb{R} \to \mathbb{R}$, integrable with respect to $F_{m,x}$. If we assume that the expected value of the random variable $Y_m^f = f(Z_{m,x})$ exists, then we can define it by the following improper Riemann-Stieltjes integral

$$E\left[f\left(Z_{m,x}\right)\right] = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}F_{m,x}(t),$$

with the requirement

$$\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}F_{m,x}(t) < \infty.$$

Therefore we can consider a general linear positive operator L_m associated with the function f and the distribution function $F_{m,x}$, defined by

(1) $$(L_m f)(x) = L_m(f(t);x) = \int_{-\infty}^{\infty} f(t)\,\mathrm{d}F_{m,x}(t).$$

It is easy to see that this is a contraction operator, since we have

$$\|L_m f\| \le \|f\| \cdot \int_{-\infty}^{\infty} \mathrm{d}F_{m,x}(t) = \|f\|.$$

Let us denote by e_r the monomial $e_r(x) = x^r$ $(r = 0,1,2, \dots)$. Beside the evident relation $L_m e_0 = e_0$, we assume that we have $L_m e_1 = e_1$ and that $L_m((t-x)^2;x) = \delta_m^2(x) \to 0$, uniformly on a compact subinterval I of the real axis. Then we can state that: *for any bounded continuous function f we have $L_m f \to f$, uniformly on I*. This result constitutes the statement of a fundamental Lemma of W. Feller ([3], pag 218), which was proved by using Chebyshev's inequality and the weak law of large numbers. Usually the operator L_m defined at (1) bears Feller's name [7], [2], [5].

It can be easily observed that this result is a consequence of the quantitative estimates of the approximation of the function f by means of the operator L_m, given in our paper [12]. In that paper we have shown how can be obtained, by using the theory of characteristic functions, the operators of Bernstein, Favard-Szasz, Baskakov, Meyer-König and Zeller and Stancu, by starting, respectively from the distributions: zero-one, Poisson, Fisher, Pascal and Markov-Polya. For the operators of interpolatory type we gave representations by means of finite differences and factorial moments of corresponding distributions. The above operators represent special cases of the Feller operators

$$(L_m f)(x) = E\left[f\left(\frac{X_1 + X_2 + \ldots + X_m}{m}\right)\right],$$

where (X_k) represents a sequence of independent random variables, identically distributed as a random variable X with finite mean x and finite variance $\delta^2(x)$. In this case $F_{m,x}$ represents the distribution function of the arithmetic mean

$$Y_m = \frac{X_1 + X_2 + \ldots + X_m}{m}.$$

Now we have

$$(L_m e_1)(x) = L_m(t;x) = x\,, \quad L_m((t-x)^2;\,x) = \delta^2(x)/m.$$

Such operators were called by R. Bojanic and M.K. Khan [2] *averaging operators*.

3. THE BETA SECOND-KIND TRANSFORM $T_{p,q}$

Let us denote by $M[0,\infty)$ the linear space of functions $g(t)$, defined for $t \ge 0$, bounded and Lebesgue measurable in each interval $[a,b]$, where $0 < a < b < \infty$.

We shall define a linear transform by using the beta distribution of second kind, with the positive parameters p and q, which has the probability density

(2) $$b_{p,q}(t) = \frac{t^{p-1}}{B(p,q)(1+t)^{p+q}},$$

where $t > 0$ and $b_{p,q}(t) = 0$ otherwise; by $B(p,q)$ is denoted the beta function. This distribution belongs to Karl Pearson's Type VI. It is easy to see that there is a considerable resemblance to the case of the gamma distribution, used by us in [12] for obtaining the Post-Widder gamma operator.

By using distribution (2) we can define the beta second-kind transform $T_{p,q}$, of a function $g \in M[0, \infty)$, by

(3) $$T_{p,q}\, g = \int_0^{\infty} g(t)\, b_{p,q}(t)\,\mathrm{d}t = \frac{1}{B(p,q)} \int_0^{\infty} g(t) \frac{t^{p-1}\,\mathrm{d}t}{(1+t)^{p+q}},$$

such that $T_{p,q} |g| < \infty$.

One observes that $T_{p,q}$ is a linear positive functional.

We need to state and prove

THEOREM 1. *The moment of order r* $(1 \le r < q)$ *of the functional* $T_{p,q}$ *has the following value*

$$(4)\qquad \nu_r(p,q) = T_{p,q}\, e_r = \frac{p(p+1)\ldots(p+r-1)}{(q-1)(q-2)\ldots(q-r)}.$$

Proof. We have

$$T_{p,q}\, e_r = \frac{1}{B(p,q)} \int_0^\infty \frac{t^{p+r-1}}{(1+t)^{p+q}}\, dt .$$

If we make the change of integration variable $y = t/(1+t)$, we have $t = y/(1-y)$, $dt = (1-y)^{-2}\, dy$ and we obtain

$$(5)\qquad T_{p,q}\, e_r = \frac{1}{B(p,q)} \int_0^1 y^{p+r-1}(1-y)^{q-r-1}\, dy = \frac{B(p+r,q-r)}{B(p,q)} .$$

Applying successively r times the known relation

$$B(a,b) = \frac{b-1}{a+b-1} B(a,b-1) ,$$

we obtain the following formula

$$(6)\qquad B(a,b-r) = \frac{(a+b-1)(a+b-2)\ldots(a+b-r)}{(b-1)(b-2)\ldots(b-r)} B(a,b) .$$

Taking $a = p + r$ and $b = q$, we get

$$(7)\qquad B(p+r,q-r) = \frac{(p+q+r-1)(p+q+r-2)\ldots(p+q)}{(q-1)(q-2)\ldots(q-r)} B(p+r,q) .$$

By using successively r times the relation

$$B(a+1,b) = \frac{a}{a+b} B(a,b),$$

we find the relation

$$B(a+r,b) = \frac{a(a+1)\ldots(a+r-1)}{(a+b)(a+b+1)\ldots(a+b+r-1)} B(a,b) .$$

If we take $a = p$ and $b = q$ we get

$$B(p+r,q) = \frac{p(p+1)\ldots(p+r-1)}{(p+1)(p+q+1)\ldots(p+q+r-1)} B(p,q)$$

By replacing it into (7) and taking (5) in consideration, we obtain the desired result (4).

4. THE FUNCTIONAL $F_m^f(p,q) = T_{p,q}(\mathscr{B}_m f)$

Now let us apply the transform (3) to the Baskakov operator $\mathscr{B}_m$, defined by

$$(8)\qquad (\mathscr{B}_m f)(t) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{t^k}{(1+t)^{m+k}} f\left(\frac{k}{m}\right).$$

We may state and prove

THEOREM 2. The $T_{p,q}$ transform of $\mathscr{B}_m f$ can expressed under the following form

$$(9)\qquad F_m^f(p,q) = T_{p,q}(\mathscr{B}_m f) =$$

$$= \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{p(p+1)\ldots(p+q-1)q(q+1)\ldots(q+m-1)}{(p+q)(p+q+1)\ldots(p+q+m+k-1)} f\left(\frac{k}{m}\right).$$

Proof. We can write successively

$$T_{p,q}(\mathscr{B}_m f) =$$

$$= \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{1}{\mathrm{B}(p,q)} \left[\int_0^\infty \frac{t^{p+k-1}}{(1+t)^{m+p+q+k}}\, dt\right] f\left(\frac{k}{m}\right).$$

If we make the change of variable $y = t/(1+t)$ in the integral

$$I_{m,k}(p,q) = \int_0^\infty \frac{t^{p+k-1}\, dt}{(1+t)^{p+q+m+k}} ,$$

we get

$$I_{m,k}(p,q) = \int_0^1 y^{k+p-1}(1-y)^{m+q-1}\, dy =$$

$$= B(k+p,m+q) = \frac{\Gamma(k+p)\Gamma(m+q)}{\Gamma(m+k+p+q)} =$$

$$= \frac{p(p+1)\ldots(p+q-1)q(q+1)\ldots(q+m-1)}{(p+q)(p+q+1)\ldots(p+q+m+k-1)} B(p,q)$$

and we obtain formula (9).

We can make the remark that if we select $p = x/\alpha$, $q = 1/\alpha$, where α is a positive parameter, then formula (9) leads us to the parameter-dependent operator $L_m^{(\alpha)}$, introduced in 1970 in our paper [14] (see also [15]), as a generalization of the Baskakov operator. By using the factorial powers, with the step $h = -\alpha$, it can be expressed under the following compact form

(10) $$\left(L_m^{(\alpha)} f\right)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^{[k,-\alpha]} \cdot 1^{[m,-\alpha]}}{(1+x)^{[m+k,-\alpha]}} f\left(\frac{k}{m}\right).$$

It should be noticed that the operator $T_{\frac{x}{\alpha},\frac{1}{\alpha}} = T^{\alpha}$ was used in the paper [1] for obtaining our operator $L_m^{(\alpha)}$ given above.

5. THE BETA SECOND-KIND LINEAR POSITIVE OPERATOR $T_{mx,\, m+1}$; APPROXIMATION PROPERTIES OF IT

Now we introduce a new beta second-kind approximating operator.

If in (3) we choose $p = mx$ and $q = m+1$, then to any function $f \in M[0,\infty)$ we associate the linear positive operator L_m, defined by

(11) $$(L_m f)(x) = T_{mx,m+1} f = \int_0^{\infty} f(t)\, b_{mx,m+1}(t)\,dt\,,$$

or more explicitly

(12) $$(L_m f)(x) = L_m(f(t);x) = \frac{1}{B(mx, m+1)} \int_0^{\infty} f(t) \frac{t^{mx-1}}{(1+t)^{mx+m+1}}\,.$$

Because

$$(L_m e_0)(x) = \int_0^{\infty} b_{mx,m+1}(t)\,dt = 1$$

and according to (4) we have $(L_m e_1)(x) = x$, it follows that L_m reproduces to linear function.

It is easily seen that this operator is of Feller's type, but it is not averaging operator.

If we use an inequality established in our paper [12] we can find immediately the order of approximation of f by means of $L_m f$.

THEOREM 3. *If $f \in C[0,\infty)$, such that $L_m(|f|;x) < \infty$, then for any $m > 1$ we have the following inequalities*

(13) $$|f(x) - (L_m f)(x)| \le \left(1 + \sqrt{x(x+1)}\right) \omega_1\left(f; \frac{1}{\sqrt{m-1}}\right),$$

(14) $$|f(x) - (L_m f)(x)| \le \left(3 + x(x+1)\right) \omega_2\left(f; \frac{1}{\sqrt{m-1}}\right),$$

where $\omega_k(f;\delta)$ represents the modulus of continuity of order k of the function f $(k = 1,2)$.

Proof. From (12) and (4) we deduce

$$(L_m e_2)(x) = \frac{x(mx+1)}{m-1} = x^2 + \frac{x(x+1)}{m-1}$$

and

$$L_m\left((t-x)^2; x\right) = \frac{x(x+1)}{m-1}.$$

If we use the following inequality, given at page 686 of our paper [12]:

$$|f(x) - (L_m f)(x)| \le (1 + \delta^{-1}\sigma_m(x))\, \omega_1(f;\delta)\,,$$

where

$$\sigma_m^2(x) = L_m\left((t-x)^2; x\right) = \frac{x+1}{m-1}$$

we obtain

$$|f(x) - (L_m f)(x)| \le \left(1 + \frac{1}{\delta} \frac{\sqrt{x(x+1)}}{\sqrt{m-1}}\right) \omega_1(f;\delta)\,.$$

If we take $\delta = 1/\sqrt{m-1}$ we arrive at inequality (13).

By using Theorem 4.1 from a paper by H. H. Gonska and J. Meier [4] we can obtain at once the next inequality (14).

According to the Bohman-Korovkin convergence criterion, we can deduce

COROLLARY 1. *If $f \in M[0,\infty)$ and is continuous at all point of an interval $[a, b]$ $(0 \le a < b < \infty)$, then $L_m f$ converges uniformly in $[a, b]$ to the function f when $m \to \infty$.*

Appealing to an inequality given at page 689 of our paper [12] we can establish an inequality of Lorentz type.

THEOREM 4. *If f has a bounded uniformly continuous derivative for $x \ge 0$, then for $m > 1$ we have*

(15) $$|f(x) - (L_m f)(x)| \le \frac{1}{\sqrt{m-1}} \cdot \sqrt{x(x+1)}\left[1 + \sqrt{x(x+1)}\right] \omega_1\left(f'; \frac{1}{\sqrt{m-1}}\right).$$

Proof. If in our mentioned inequality

$$|f(x) - (L_m f)(x)| \le \sigma_m(x)\left[1 + \sigma_m(x)\right] \omega_1(f';\delta)$$

we replace

$$\sigma_m(x) = \sqrt{x(x+1)}\,/\sqrt{m-1}\,,\quad \delta = 1/\sqrt{m-1}$$

we arrive just to the inequality (15).

For the operator L_m introduced at (12) one can be established also an asymptotic formula of Voronovskaja type.

THEOREM 5. *If $f \in M[0,\infty)$ is differentiable in some neighborhood of a point $x \in [a,b]$ $(0 < a < b < \infty)$ and at this point the second derivative exists, then we have*

$$(16) \qquad \lim_{m\to\infty} m\left[f(x)-(L_m f)(x)\right] = -\frac{x(x+1)}{2} f''(x).$$

If $f \in C^2[a,b]$ then the convergence is uniform.

For the proof of this theorem we can use a result of R. G. Mamedov [9] (see also M.W. Müller [11]).

Concerning the remainder of the approximation formula

$$(17) \qquad f(x) = (L_m f)(x) + (R_m f)(x),$$

which has the degree of exactness $N=1$, we can give an integral representation.

THEOREM 6. *If the function $f \in M[0,\infty)$ has a continuous second derivative for $t \geq 0$, then we can represent the remainder of formula* (17) *in the following integral form*

$$(18) \qquad (R_m f)(x) = \int_0^\infty G_m(t;x) f''(t)\,dt,$$

where

$$G_m(t;x) = (R_m \varphi_x)(t), \ \varphi_x(t) = (x-t)_+ = \frac{x-t+|x-t|}{2}$$

and R_m operates on $\varphi_x(t)$ as a function of x.

This representation can be obtained at once if we apply the well-known theorem of Peano.

It can be easily verified that for any fixed point $x \in (0,\infty)$ we have $G_m(t;x) \leq 0$ when $t \in [0,\infty)$. Consequently we may apply the mean value theorem of the integral calculus and we obtain

$$f(x) = (L_m f)(x) + f''(\xi)\int_0^\infty G_m(t;x)\,dt.$$

If we choose $f(x) = e_2(x) = x^2$, we can deduce from this equality the value of the integral of Peano's kernel

$$\int_0^\infty G_m(t;x)\,dt = -\frac{x(x+1)}{2(m-1)}.$$

Therefore from the preceding theorem there follows

COROLLARY 2. *If $f \in C^2[0,\infty)$ then for all $m > 1$ there exists a point $\xi \in (0,\infty)$ such that the remainder of the approximation formula* (17) *can be represented under the following form*

$$(19) \qquad (R_m f)(x) = -\frac{x(x+1)}{2(m-1)} f''(\xi), \ \xi \in (0,\infty).$$

Finally, we mention that our operator defined at (11) – (12) is distinct from other beta operators considered earlier in the papers [10], [8], [17], [6], [1].

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