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ON THE BEHAVIOUR OF THE TANGENTIAL MODULUS OF A BANACH SPACE I

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1. INTRODUCTION AND NOTATION

In some investigations on Banach spaces and their applications it is sometimes useful to know the geometry of the unit balls. The geometry of the balls may be reflected in the behaviour of some moduli, i.e. of some real functions attached to a Banach space.

In this paper the properties of such a modulus are discussed. The invoked modulus has been recently introduced and used (see[7]) at existence problems for the Lipschitz continuous selections of set-valued mappings. A new geometric definition of this modulus is given. For some reasons (see Proposition 2.6) it will be called the *tangential modulus*.

From the behaviour of the tangential modulus in the neighbourhood of some points we obtain information about the geometry of the Banach spaces. A characterization of the uniform convexity of a Banach space is reconsidered. The convexity of the tangential modulus in the neighbourhood of 1 and connections with known moduli is presented too.

Let $(X, \|\cdot\|)$ be a real Banach space and let X^* be its dual. To avoid trivialities we assume that X has dimension at least two. For $r > 0$ and $x \in X$ denote by $B(x,r)$ the closed ball with center x and radius r and by $B(X) = B(X,\|\cdot\|)$ the unit ball of X. Analogously, $S(X)$ will represent the unit sphere of X. Choosing $x, y \in X$, $x \neq y$ we shall consider the straight line passing through x and y denoted by xy as well as the open and the closed line segment with the vertices x and y denoted by $(x; y)$ respectively by $[x; y]$. Let $x,y,z \in X$, $x \neq y$. A parallel to xy from z is the set $\{z + \lambda(x-y) : \lambda \in R\}$. The symbol $\perp_{BJ}$ will be used for Birkhoff-James's orthogonality in $(X,\|\cdot\|)$; namely: $x \perp_{BJ} y$ if $\|x\| \leq \|x + \lambda y\|$, for all $\lambda \in \mathbf{R}$.

The *modulus of smoothness* of X is defined [5] by:

$$\rho_X(\tau) = \rho(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x-\tau y\| - 2) : x, y \in S(X)\right\}, \quad \tau \geq 0$$

and the *modulus of convexity* [5] by:

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x, y \in S(X), \|x-y\| = \varepsilon\right\}, \quad 0 \leq \varepsilon \leq 2.$$

The Banach space X is said to be *uniformly convex* if $\delta_X(\varepsilon) > 0$, $0 < \varepsilon \leq 2$, and *uniformly smooth* if $\lim_{\tau\to 0} \rho_X(\tau)/\tau = 0$.

The *orthogonal modulus of smoothness* (see T. Figiel [3], p. 129) is the function $\overline{\rho}_X$ defined by

$$\overline{\rho}_X(\tau) = \sup\left\{\frac{1}{2}(\|x+\tau y\| + \|x-\tau y\| - 2) : \ x, y \in S(X), \text{ and}\right.$$

$$\left.\text{there exists } x^* \in S(X^*) \text{ such that } x^*(x) = 1,\ x^*(y) = 0\right\}, \quad r \geq 0.$$

It is well known (see D. Amir [1], p. 33) that the condition of orthogonality used in the definition of $\overline{\rho}_X$ is equivalent to Birkhoff-James's orthogonality.

For any $x,y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x,y)$ in $(x; y)$ with $\|z\| = 1$. We put as in [7]

$$\omega(x,y) = \frac{\|x - z(x,y)\|}{\|x\| - 1},$$

and define the function $\xi = \xi_X : [0,1) \to R$ by

$$\xi(\beta) = \sup\{\omega(x,y) : \|y\| \leq \beta < 1 < \|x\|\}, \quad 0 \leq \beta < 1.$$

K. Przeslawski and D. Yost name this function in two different ways (in the two variants of the preprint [7]).

2. PRELIMINARY RESULTS

In the sequel the following simple geometrical lemma will be frequently applied.

LEMMA 2.1. *If* $x \perp_{BJ} y$ *and* $0 < a < b$, *then* $\|x + ay\| \leq \|x + by\|$.

Proof. From the Birkhoff- James's orthogonality we have $\|x\| \leq \|x + ay\|$. If $\|x\| = \|x + ay\|$ then $x + ay \perp_{BJ} y$ and the result follows. If $\|x\| < \|x + ay\|$ then the collinear points $x, x+ay, x+by$, $0 < a < b$ are in this order in the interior, on the boundary, respectively in the exterior of $B(0, \|x + ay\|)$. □

One observes that symmetrically if $x \perp_{BJ} y$ and $b < a < 0$ then also $\|x + ay\| \leq \|x + by\|$.

The following useful result can be find in [2], [8], [4].

LEMMA 2.2. *Let X be a two-dimensional Banach space and let K_1, K_2 be closed convex subsets of X with nonvoid interiors. If $K_1 \subseteq K_2$ then $r(K_1) \leq r(K_2)$, where $r(K_i)$ denotes the length of the circumference of K_i, $i = 1, 2$.*

Now, we present a first result with respect to the function ξ.

LEMMA 2.3. *Let X be a two-dimensional Banach space and $x, y \in X$ be such that*

$$\|y\| \leq \beta < 1 < \|x\|.$$

Then there exists a vector $y' \in B(0, \beta)$ such that xy' is the supporting line of $B(0, \beta)$ and

$$\omega(x, y) \leq \omega(x, y').$$

Proof. If x and y are linearly dependent, then $\omega(x, y') \geq \omega(x, y) = 1$, for every $y' \in B(0, \beta)$. In the other case, in every semi-plane determined by $0x$ there exists a supporting line of $B(0, \beta)$ passing through x. Let xy', $(y' \in B(0, \beta))$ be the supporting line of $B(0, \beta)$ contained in the semi-plane determined by $0x$ and y. Then the triangle with vertices $0, x, z(x, y)$ is contained in the triangle determined by $0, x, z(x, y')$. From Lemma 2.2 we have:

$$\|z(x,y)\| + \|x - z(x,y)\| + \|x\| \leq$$
$$\leq \|z(x,y')\| + \|x - z(x,y')\| + \|x\|,$$

and hence $\omega(x, y) \leq \omega(x, y')$. □

Remark 2.4 From Lemma 2.3 the function ξ may be defined by

$$\xi(\beta) = \sup\{\omega(x,y) : \|y\| = \beta, \|x\| > 1, \ y \perp_{BJ} (x - y)\}, \quad \beta \in [0,1).$$

LEMMA 2.5. *Let X be a two-dimensional Banach space and let x,y be in X such that*

$$\|y\| \leq \beta < 1 < \|x\|.$$

If $x' \in (x; z(x, y))$ then

$$\omega(x, y) \leq \omega(x', y).$$

Proof. We have $\omega(x, y) = \omega(x', y) = 1$ for x and y two collinear vectors. Let x and y be linearly independent. Denote by x_1 the projection $x / \|x\|$ of x on $B(X)$; analogously $x'_1 = x' / \|x'\|$. It is clear that $z(x, y) = z(x', y) = z$. The parallel to the straight line xy from origin intersects the straight lines zx_1, respectively zx'_1 in x_2 respectively x'_2. A comparison of similar triangles x_1zx and x_1x_20 yields

$$\frac{\| x - z(x,y) \|}{\| x \| - \| x_1 \|} = \frac{\| x_2 \|}{\| x_1 \|}$$

and then $\omega(x, y) = \| x_2 \|$. By a similar argument one obtains $\omega(x', y) = \| x'_2 \|$. From the convexity of $B(X)$ it follows that $\| x'_2 \| \geq \| x_2 \|$. □

PROPOSITION 2.6. *The function* ξ *can be defined by*

$$\xi(\beta) = \sup\{ \| u - v \| : \ u \in S(X), v \in X, u \perp_{BJ} v, \quad \min_{\lambda \geq 0} \| (1-\lambda)u + \lambda v \| = \beta \}, \tag{1}$$

for each $\beta \in [0,1)$, *namely* $\xi(\beta)$ *represents the maximal length of the line segments* $[u; v]$, *uv being a support line of the ball* $B(0,\beta)$ *while* $u \in S(X)$ *and* $u \perp_{BJ} v$.

Proof. Applying remark 2.4 it is sufficient to suppose (in $\omega(x, y)$), $\| y \| = \beta$ and that xy is a supporting line of the ball $B(0,\beta)$. Then using Lemma 2.5 and the corresponding notation one gets:

$$\xi(\beta) \leq \sup\{ \| x_2 \| : z(x,y) \perp_{BJ} (z(x,y) - x_2), \quad y \perp_{BJ} (y - z(x,y)) \}, \quad 0 < \beta < 1. \tag{*}$$

In fact in (*) instead of an inequality we have an equality as we can see a little later. Now, if the parallel to the support line x_2z of $B(X)$ from origin intersects the straight line xy in v and if we write $u = z(x, y)$, then from the parallelogram $uv0x_2$ we have $\| x_2 \| = \| u - v \|$, $\| u \| = 1$, $u \perp_{BJ} v$, $\min_{\lambda \geq 0} \| (1-\lambda)u + \lambda v \| = \beta$, and formula (1) follows. It remains to prove the reverse inequality in (*). Let z be in $S(X)$ and d be a supporting line of $B(X)$ passing through z. Suppose $y \in B(0, \beta)$ and zy is a support line of $B(0, \beta)$. Denote by z_1 the vector $(1 + \varepsilon) z$ with $\varepsilon > 0$ fixed and by z_2 the intersection of $(y; z_1)$ with $S(X)$. In the two-dimensional space spanned by y and z the parallel to yz_1 (respectively to yz) from origin intersects the straight line zz_2 (respectively d) in x'_2 (respectively x_2). Then $\omega(z_1, y) = \| x'_2 \|$. If $\varepsilon \searrow 0$ then x'_2 tends to a vector x''_2 collinear to x_2. By the convexity of $B(X)$ we have $\| x''_2 \| \geq \| x_2 \|$. □

The new definition of ξ_X enables us to call now this function the *tangential modulus* of X. In [7] the authors have presented some applications and essential properties of the tangential modulus. For instance, it was proved that ξ is an increasing function, $\xi(0) = 1$,

$$\xi_X(\beta) \leq (1+\beta)/(1-\beta) = \xi_{l^1(2)}(\beta), \ \ \beta \in [0,1)$$

and that if H is a Hilbert space then $\xi_H(\beta) = (1 - \beta^2)^{-1/2}$. The locally Lipschitz continuity of ξ was obtained by the sharp inequality

$$\xi_X(\gamma) - \xi_X(\beta) \leq \xi_{l^1(2)}(\gamma) - \xi_{l^1(2)}(\beta) = \frac{2(\gamma - \beta)}{(1-\beta)(1-\gamma)},$$

$0 \leq \beta \leq \gamma < 1$. The geometry of the unit ball of X was reflected in the behaviour of the function ξ. For instance, one obtains that X is uniformly convex if and only if $\liminf_{\beta \nearrow 1} (1 - \beta)\xi_X(\beta) = 0$.

3. THE BEHAVIOUR OF THE TANGENTIAL MODULUS IN THE NEIGHBOURHOOD OF 1

In the sequel we shall continue the investigation of the properties of ξ insisting on the relations between the behaviour of the function ξ and the geometry of the unit sphere of X.

First of all, one observes that $\lim_{\beta \to 1} \xi_X(\beta) = \infty$. For the proof it is sufficient to consider only the two-dimensional spaces. Let F be a two-dimensional space and let u be a point of smoothness of the unit sphere $S(F)$. Denote by d the support line of $B(F)$ passing through u and by d_1 a parallel to d from origin. Choose $v_n \in d_1$ such that $\| v_n \| = n \in N$. Then the straight line uv_n contains at least a point t_n with $\| t_n \| = \beta_n < 1$. We have that $\xi_X(\beta_n) \geq n$ and since ξ_X is increasing it follows that $\lim_{\beta \to 1} \xi_X(\beta) = \infty$.

On the other hand if β is chosen so closed to 1 that $\xi_X(\beta) > 1 + \beta$ and if the vectors u and v in formula (1) verify $\| u - v \| > 1 + \beta$ then $\min_{\lambda \geq 0} \| (1-\lambda)u + \lambda v \|$ is attained for $\lambda \in (0,1)$. Indeed, for $t \in S(0,\beta)$ and $t = \lambda v + (1 - \lambda)u$, with $\lambda \geq 1$ we have

$$\| u - v \| \leq \| u - t \| \leq \| u \| + \| t \| = 1 + \beta,$$

which is impossible. It is clear that $\xi_{l^1(2)}(\beta) > 1 + \beta$, for all $\beta \in (0,1)$ and so in this case λ in formula (1) can be taken in $[0,1]$. In the opposite case when X is a Hilbert space we have that if $t \in S(0,\beta)$ and $t \perp (u - t)$ then from the orthogonality's symmetry it follows: $\| u - t \| < \| u \| = 1$. Since $\xi_X(\beta) \geq 1$, for all $\beta \in (0,1)$ it implies that λ in formula (1) can be taken in $[0,1]$. I do not know if λ in formula (1) can be taken only in $[0,1]$ for every Banach space X and every $\beta \in (0,1)$.

Now we compute again the tangential modulus $\xi_H(\beta)$ where H is a Hilbert space and $(\cdot \mid \cdot)$ denotes its inner product. In this case we have:

$$\xi_H(\beta) = \| u - v \|; \text{ with } u, v \in H \text{ such that } u \in S(H),$$
$$(u \mid v) = 0 \text{ and } \min_{\lambda \in [0,1]} \| \lambda u + (1-\lambda) v \| = \beta.$$

It means that for Hilbert spaces the "sup" in formula (1) can be omitted. Indeed, let u, v be as above. Then

$$\min_{\lambda \in [0,1]} \| \lambda u + (1-\lambda) v \|^2 =$$
$$= \min_{\lambda \in [0,1]} \left[\lambda^2 \left(1 + \| v \|^2\right) - 2\lambda \| v \|^2 + \| v \|^2 \right] = \| v \|^2 / \left(1 + \| v \|^2\right) = \beta^2 .$$

We have $\| v \| = \beta(1-\beta^2)^{-1/2}$ and

$$\| u - v \| = \left(1 + \| v \|^2\right)^{1/2} = \left(1 - \beta^2\right)^{-1/2} = \xi_H(\beta) .$$

PROPOSITION 3.1. *For every Banach space X the tangential modulus ξ_X is a convex function in a neighborhood of 1.*

Proof. From the continuity of ξ it is sufficient to prove that:

$$\xi\left(\frac{\beta+\gamma}{2}\right) \le \frac{1}{2}\left(\xi(\beta) + \xi(\gamma)\right), \quad \beta_0 \le \beta < \gamma < 1 ,$$

where β_0 is chosen so that $\xi(\beta_0) > 2 > 1 + \beta_0$. Let $u, v \in X$ be such that $\xi((\beta+\gamma)/2) \le \| u - v \| + \varepsilon$, $\varepsilon > 0$ being arbitrarily small. Here $\| u \| = 1$, $u \perp_{BJ} v$, $\min_{\lambda \in [0,1]} \| \lambda u + (1-\lambda) v \| = (\beta+\gamma)/2 = \| y \|$, y being in $[u,v]$ and $\beta_0 \le \beta < \gamma < 1$. Now we consider the collinear vectors $y_\beta = 2\beta y / (\beta+\gamma) \in B(0,\beta)$ and $y_\gamma = 2\gamma y / (\beta+\gamma) \in B(0,\gamma)$. Let v_β respectively v_γ be defined by: $v_\beta = u y_\beta \cap 0v$ respectively $v_\gamma = u y_\gamma \cap 0v$. The parallel to $0y$ from v_β intersects uv respectively uv_γ in z respectively z_γ and the parallel to uv from z_γ intersects $0v$ in w_γ. It is clear that z is the middle point of $[v_\beta; z_\gamma]$ and v is the middle point of $[v_\beta; w_\gamma]$. We have :

$$\xi\left(\frac{\beta+\gamma}{2}\right) \le \| u - v \| + \varepsilon = \left\| \frac{u - v_\beta + u - w_\gamma}{2} \right\| + \varepsilon \le$$
$$\le \frac{1}{2}\left(\| u - v_\beta \| + \| u - w_\gamma \| \right) + \varepsilon .$$

Since $w_\gamma \in [v; v_\gamma]$ and $u \perp_{BJ} v$, by Lemma 2.1 one obtains:

$$\| u - w_\gamma \| \le \| u - v_\gamma \| ,$$

and

$$\xi\left(\frac{\beta+\gamma}{2}\right) \le \frac{1}{2}\left(\| u - v_\beta \| + \| u - v_\gamma \| \right) + \varepsilon \le \frac{1}{2}\left(\xi(\beta) + \xi(\gamma)\right) + \varepsilon , \ \varepsilon > 0$$

and the convexity of ξ follows. □

As it is well known (see V.I. Liokoumovich [6]) the modulus of convexity δ_x is not always a convex function, but it is a simple exercise to prove the convexity of modulus of smoothness ρ_x and the convexity of the orthogonal modulus of smoothness $\overline{\rho}_X$. Now, because the tangential modulus is convex in a neighbourhood of 1 we can expect that there exists a strong relation between ξ_X and ρ_X, (respectively $\overline{\rho}_X$.) Such a subject will be treated elsewhere in the second part of this paper. There, the behaviour of the tangential modulus in the neighbourhood of 0 is crucial. So, it is natural to study also the behaviour of ξ_X in the neighbourhood of 1. In this direction the following proposition was proved in the second variant of the preprint [7]. In the spirit of this paper we give finally a new proof of the "if" part.

PROPOSITION 3.2. *The Banach space X is uniformly convex if and only if*

$$\lim\ \inf_{\beta \nearrow 1} (1-\beta)\xi_X(\beta) = 0 .$$

Proof. Suppose that X is uniformly convex and $\lim\ \inf_{\beta \nearrow 1}(1-\beta)\xi_X(\beta) = 0$. Let $u \in S(X)$, $v \in X$ be such that $u \perp_{BJ} v$ and $\min_{\lambda \ge 0} \| \lambda v + (1-\lambda) u \| = \beta$. Let $y \in uv \cap B(0,\beta)$ and let w be the second intersection of uv with $B(X)$. The segment line $[0; u]$ intersects $B(0, \beta)$ in u_1. In the two-dimensional space generated by u and v the parallel to $0v$ from u_1 intersects uv in v_1. It is clear that $v_1 \in [y; u]$. A comparison of similar triangles uu_1v_1 and $u \circ v$ yields:

$$\frac{\| u - u_1 \|}{\| u \|} = \frac{\| u - v_1 \|}{\| u - v \|},$$
$$(1-\beta) \| u - v \| = \| u - v_1 \| .$$

Passing to supremum over all pairs (u, v) with $\| u \| = 1$, $u \perp_{BJ} v$ and $\min_{\lambda \ge 0} \| \lambda v + (1-\lambda) u \| = \beta$ we have

$$(1-\beta)\xi(\beta) = \sup \| u - v_1 \| \le \sup \| u - w \| .$$

For every β sufficiently close to 1, there exists a pair (u,v) such that $\| u - w \| \ge b/2$, $\| u \| = \| w \| = 1$ and from $y \perp_{BJ} (u - w)$ it follows:

$$1-\left\|\frac{u+w}{2}\right\|\leq 1-\|y\|=1-\beta$$

and so $\delta_X(b/2)=0$, a contradiction with the uniform convexity of X. □

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