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BOOK REVIEWS

MARIO MILMAN, *Extrapolation and Optimal Decompositions, (with Applications to Analysis)*, Lecture Notes in Mathematics, vol. 1580, Springer-Verlag, Berlin-Heidelberg-New York, 1994, 60pp.

The extrapolation theory, elaborated by the author, mainly in collaboration with B. Jawerth (see Björn Jawerth and Mario Milman, Extrapolation Theory with Applications, Memoirs Amer. Math. Soc. vol. 440 (1991)) is concerned with a problem which is somehow converse to the interpolation problem.

The basic notion of interpolation theory is that of Banach couple, which means a pair $\overline{A} = (A_0, A_1)$ of Banach spaces embeded in a Hausdorff topological vector space H. Let $\Delta(\overline{A}) = A_0 \cap A_1$ and $\Sigma(\overline{A}) = A_0 + A_1$ equipped with the norms $\|x\|_\Delta = \max\{\|x\|_i : i = 0,1\}$ and $\|x\|_\Sigma = \inf\{\|x_0\|_{A_0} + \|x_1\|_{A_1} : x = x_0 + x_1,\ x_i \in A_i,\ i = 0,1\}$, respectively. An intermediate space is a Banach space A such that $\Delta(\overline{A}) \to A \to \Sigma(\overline{A})$. The spaces A and B are called interpolation spaces with respect to the couples $\overline{A} = (A_0, A_1)$ and $\overline{B} = (B_0, B_1)$ if $T : \overline{A} \to \overline{B}$ implies $T : A \to B$. If moreover $\|T\|_{A,B} \le \max(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1})$ then A, B are called exact interpolation spaces. An interpolation method is a functor F defined on the category of Banach couples and linear bounded operators between them, such that $F(\overline{A})$, $F(\overline{B})$ are interpolation spaces for $\overline{A}$, $\overline{B}$ and $F(T) = T$ for all $T: \overline{A} \to \overline{B}$. The interpolation method is called exact if it yields exact interpolation spaces. A good reference for the interpolation theory, both classical and abstract, is Yu. A. Brudnyi and N. Ya. Krugljak, Interpolation Functors and Interpolation Spaces, vol. I, North-Holland Math. Library vol. 47, 718 pp., Amsterdam New York, Oxford, Tokyo, 1991.

The extrapolation theory is dealing with the converse problem: Given a family of interpolation spaces reconstruct the originating pair. In this formulation, the problem is directly related to best possible interpolation theorems and in some sense, it could be considered as a chapter of interpolation theory of infinitely many spaces. The precise connection between these theories is an open problem.

The book is dealing also with weaker formulations of the problem, such as the extrapolation of the continuity of an operator T or the extrapolation of inequalities for its norm, usually based on the basic functionals K and J.

More exactly, let $\{A_\theta : \theta \in \Theta\}$ be a family of Banach spaces indexed by some fixed index set Θ (usually $\Theta = (0,1)$). These families of Banach spaces are strongly compatible in the sense that there are two Banach spaces Δ and Σ (depending on the family $\{A_\theta\}$) such that $\Delta \subset A_\theta \subset \Sigma$, $\theta \in \Theta$. If $\{A_\theta\}$ and $\{B_\theta\}$ are two families of strongly compatible Banach spaces, $\Delta_a \subset A_\theta \subset \Sigma_a$, $\Delta_b \subset B_\theta \subset \Sigma_b$, a natural morphism is a bounded linear operator $T : \{A_\theta\} \to \{B_\theta\}$, i.e. $T : \Sigma_a \to \Sigma_b$ is an operator whose restrictions to A_θ maps A_θ into B_θ with norm ≤ 1, $\theta \in \Theta$.

Two Banach spaces A and B are called extrapolation spaces for $\{A_\theta\}$ and $\{B_\theta\}$ if $\Delta_a \subset A \subset \Sigma_a$, $\Delta_b \subset B \subset \Sigma_b$ and T : $\{A_\theta\} \xrightarrow{\quad 1 \quad} \{B_\theta\}$ implies T : $A \xrightarrow{\quad 1 \quad} B$.

An extrapolation method E is a functor defined on a collection dom (E) of families of strongly compatible spaces such that $E\{A_\theta\}$ and $E\{B_\theta\}$ are extrapolation spaces for $\{A_\theta\}$ and $\{B_\theta\}$. The simplest extrapolation methods are the Σ and Δ methods.

Chapter 2, Background on Extrapolation Theory, contains a detailed introduction to abstract extrapolation theory. Chapter 1, Introduction, contains a brief guide to interpolation theory with references to the basic textbooks in the field.

The presentation of the theory is oriented towards applications to classical analysis and the contents of the book reflects the interplay between abstract and classical analysis. As points out the author in the preface the abstract theory provides a framework for consolidation, extension and simplification of results in classical analysis. This is the case even with S.Yano's classical paper (J.Math.Soc.Japan 3(1951), 296-305), which can be considered as the starting point of the extrapolation theory, although some special cases had been considered earlier by Titchmarsh and Marcinkiewicz and are included in Zygmund's treatise on trigonometric series. The abstract methods of extrapolation theory yield sharper versions even in the p setting of Yano's theorem.

The development of the theory is arranged in close connection with its applications to various branches of analysis. Each section is to a large extent independent of the others so that a prospective reader, interested in some specific topics, may skip the irrelevant parts. At the same time the author tried to avoid too much repetition.

The book is divided into ten chapters:

1. Introduction (containing a brief presentation and guide to the literature on interpolation theory); 2. Background on the Extrapolation Theory; 3. K/J Inequalities and Limiting Embedding Theorems; 4. Calculations with Δ Method and Applications; 5. Bilinear Extrapolation and a Limiting Case of a Theorem of Cwickel; 6. Extrapolation, Reiteration and Applications; 7. Estimates for Commutators in Real Interpolation; 8. Sobolev Imbedding Theorems and Extrapolation of Infinitely Many Operators; 9. Some Remarks on Extrapolation Spaces and Abstract Parabolic Equations; 10. Optimal Decompositions, Scales, and Nash-Moser Iteration.

The book is clearly written and contains new material appearing for the first time in print. Numerous open problems scattered through the book make it very interesting for those wanting to work in this field. The book supplies those interested in applications to various branches of analysis such as partial differential equations, approximation theory, harmonic analysis, numerical analysis, operator theory, Sobolev spaces, etc., with a fine and powerful instrument of investigation.

As the author mentions in the Introduction, these Notes are intended to serve as a basis for a larger, more detailed and formal book.

We recommend warmly these Notes to all people working in various branches of abstract or classical analysis.

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