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VARIATIONAL APPROXIMATION FOR A DIRICHLET-NEUMANN PROBLEM OF THE HEAT CONDUCTION THROUGH RECTANGULAR PLATES

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1. BASIC EQUATIONS

a) *The boundary value problem.* The differential problem with non-homogeneous mixed boundary conditions (Dirichlet-Neumann) is considered on the rectangular domain $\Omega = (0, a) \times (0, b)$, with the boundary $\partial\Omega$, with respect to the unknown function U (Fig. 1):

$$(1.1)\qquad LU \equiv -\left(\lambda_1 \frac{\partial^2 U}{\partial x^2} + \lambda_2 \frac{\partial^2 U}{\partial y^2}\right) = f_1(x, y),\ (x, y) \in \Omega$$

$$U(x, y) = 1 \quad \text{on} \quad \partial\Omega_1 \equiv OA$$

$$\frac{\partial U}{\partial N} = 0 \quad \text{on} \quad \partial\Omega_2' \equiv \overline{OC} \cup \overline{CB}$$

$$\frac{\partial U}{\partial N} = -1 - g_1(y) \quad \text{on} \quad \partial\Omega_2'' \equiv \overline{AB}$$

where: $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$, $\partial\Omega_2 = \partial\Omega_2' \cup \partial\Omega_2''$; $1 + g_1(y) \equiv g(y)$, f_1 are given functions $\left[g_1(0) = 0,\ g_1'(b) = 0\right]$; λ_i (const.) > 0 are given and $\partial U/\partial N$ is the derivative along the conormal on $\partial\Omega$. The boundary value problem models either heat conduction in a plate (if U is the temperature in a rectangle of conducting material with one edge at unit temperature, two edges thermal insulated and the fourth having a heat flux; f_1 - source function of the heat) or the flow of an inviscid fluid (if U is the velocity potential, $\lambda_i = 1, f_1 = 0$).

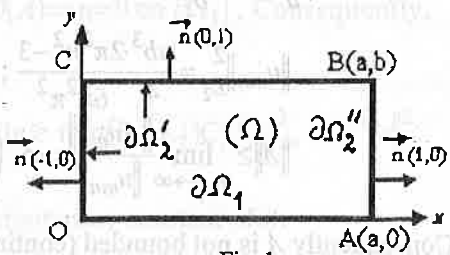

Fig. 1

b) *Operatorial equation (homogeneous boundary conditions)*. The boundary differential problem can be written in the operational form:

$$Au = f,\ f \in H \tag{1.2}$$

where

A: $H \to H$ ($H = L_2(\Omega)$) - Hilbert space) is a linear differential operator, with

$$D(A) = \left\{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \,\middle|\, Au \in L_2(\Omega),\ u = 0 \text{ on } \partial\Omega_1,\ \frac{\partial u}{\partial N} = \vec{n}^T \cdot Q\nabla u = 0 \text{ on } \partial\Omega_2 \right\}$$

(1.3)
$$Au = -\nabla \cdot (Q\nabla u),\ Q = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\ \nabla = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix},\ \vec{n} = \begin{Bmatrix} n_1 = \cos(\vec{n}, x) \\ n_2 = \cos(\vec{n}, y) \end{Bmatrix}$$

$$u = U - (1 + \psi),\ \psi(x,y) = h(x,y)\frac{\omega(x,y)}{|\nabla\omega|} - \frac{1}{\lambda_1}\frac{x^2}{2a}g_1(y),$$

$$h(x,y) = \frac{1}{\lambda_1}\frac{x(b-y)}{ab - xy},\ \omega(x,y) = xy(a-x)(b-y);\ \nabla\omega \neq 0 \textit{ on } \partial\Omega_1,\ \partial\Omega_2',\ \partial\Omega_2'';$$

$$f(x,y) = f_1(x,y) + \lambda_1\frac{\partial^2\psi}{\partial x^2} + \lambda_2\frac{\partial^2\psi}{\partial y^2}.$$

Remarks. 1°. The function ψ has been determined as follows: first, the function ω ($\omega = 0$ on $\partial\Omega$) is defined, the normalized function $\omega/|\nabla\omega|$ is introduced (in the sence of the R-functions theory, [4]) and then ψ is calculated as shown in (1.3).

2°. From a theoretical point of view, the introduction of the homogeneous boundary conditions is a useful result. In this case, the definition domain $D(A)$ of operator A is a linear subspace of space $H = L_2(\Omega)$ and the theory of linear operators on the Hilbert space H can be employed in order to study equation (1.2).

c) *The properties of the operator* $A : D(A) \subset L_2(\Omega) \to L_2(\Omega)$.

1. Operator A is unbounded on $D(A)$ [$D(A)$ is dense in $L_2(\Omega)$]. This is verified by considering the function sequence $(u_{mn}) \subset D(A)$: $u_{mn}(x, y) = (b-y)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $(x, y) \in \Omega$; $m,n = 1,2,3,\ldots$ We obtain

$$\|u_{mn}\|^2_{L_2} = \frac{ab^3}{4}\frac{2\pi^2 n^2 - 3}{6n^2\pi^2};$$

$$\|A\| \geq \lim_{m,n\to\infty}\frac{\|Au_{mn}\|}{\|u_{mn}\|} = \infty,\ \left(\|Au_{mn}\|_{L_2} \to \infty \text{ for } m,n \to \infty\right)$$

Consequently A is not bounded (continuous) [3], [7].

2. The operator A is pre-closed (it possesses closed extension).

Indeed, with $\varphi \in C_0^\infty(\Omega)$, $v_{mn} (\in D(A)) \to 0$ and $(Av_{mn}) \to w \in L_2(\Omega)$ we have

$$(Av_{mn}, \varphi)_{L_2(\Omega)} = -\iint_\Omega v_{mn}\nabla \cdot Q\nabla\varphi \,dx\,dy \quad (\to 0 \text{ for } m,n \to \infty) \tag{1.4}$$

$$\lim_{m,n\to\infty}(Av_{mn}, \varphi)_{L_2(\Omega)} = (w, \varphi)_{L_2(\Omega)} = 0 \Rightarrow w \perp \varphi,\ \forall\varphi \in C_0^\infty(\Omega) \tag{1.5}$$

$$\Rightarrow w = 0 \quad \left(\text{the set } C_0^\infty(\Omega) \text{ is dense in } L_2(\Omega)\right).$$

This proves (in accordance with the definition) that A is pre-closed, [7].

3. The operator A is symmetrical. For the operator A (unbounded) we have

1°) $D(A)$ is dense in $H = L_2(\Omega)$

$$2°)\ \forall u, v \in D(A),\ (Au, v)_{L_2} = \iint_\Omega \nabla^T v \cdot (Q\nabla u)\,dx\,dy = \tag{1.6}$$
$$= -\iint_\Omega u\nabla \cdot (Q\nabla v)\,dx\,dy = (u, Av)_{L_2}$$

4. The symmetrical operator A is positive definite on $D(A)$, i.e. $\exists\alpha^2$ (const.) > 0 so that $(Au,u)_{L_2(\Omega)} \geq \alpha^2(u,u)_{L_2(\Omega)}$, $\forall u \in D(A)$.

Proof. From (1.6) we obtain

$$(Au,u)_{L_2(\Omega)} = \iint_\Omega \nabla^T u \cdot Q\nabla u\,dx\,dy \tag{1.7}$$

Now, the Friedrichs generalized inequality is applied. According to this, [2], for a domain Ω that a boundary $\partial\Omega$ with part of it $\partial\Omega_1$ (open) of Lebesgue positive measure $m(\partial\Omega_1) > 0$, there exists a constant $C_F > 0$ (depending only of Ω and $\partial\Omega_1$) so that

$$\forall u \in H^1(\Omega),\ \|u\|^2_{H^1(\Omega)} \leq C_F\left[\iint_\Omega |\nabla u|^2 dx\,dy + \int_{\partial\Omega_1} u^2 ds\right] \tag{1.7}$$

where $H^1(\Omega)$ is Sobolev space and $\|u\|_{H^1} \geq \|u\|_H$. If $u \in D(A)$ we have $u \in H^1(\Omega)$ and (1.7) can also be applied on $D(A)\left[u \in D(A) \Rightarrow u = 0 \text{ on } \partial\Omega_1\right]$. Consequently, $\forall u \in D(A)$,

$$(Au,u)_{L_2} \geq \min(\lambda_1, \lambda_2)\iint_\Omega |\nabla u|^2 dx\,dy \geq \min(\lambda_1, \lambda_2)C_F^{-1}\|u\|^2_{H^1} \geq \alpha^2\|u\|^2_{L_2} \tag{1.8}$$

where $\alpha^2 = \frac{1}{C_F}\min(\lambda_1, \lambda_2)$ is the positive definiteness constant of A.

2. APPLICATION OF THE RITZ VARIATIONAL METHOD TO PROBLEM (1.1)

a) *The variational functional and the variational formulation associaed to the boundary value problem.* The energy variational functional F, which has a minimum value on the solution of (1.2), can be conected to the operatorial equation (1.2) [according to the properties of operator A defined on $D(A)$ – dense in $H = L_2(\Omega)$ $(\equiv L_2)$]:

(1°) $F(u)=(Au,u)_{L_2}-2(f,u)_{L_2}, u\in D(A); \left(f\in C(\overline{\Omega})\right)$

or

(2°) $F(u)=\int\int_{\Omega}\nabla^T u\cdot Q\nabla u\,dxdy-2\int\int_{\Omega} fu\,dxdy$

$u\in D(F)=D_0(F)$ or $u\in D(F)=H^1_{01}(\Omega)$

where $D_0(F)=\left\{u\in C^2(\Omega)\cap C(\overline{\Omega})\,\middle|\, u=0 \text{ on } \partial\Omega_1\right\}$ and $H^1_{01}(\Omega)$ is the linear space

$$H^1_{01}(\Omega)=\left\{u\in H^1(\Omega)\,\middle|\,u=0 \text{ on } \partial\Omega_1\right\}$$

with the norm

$$\|u\|_{H^1_{01}}=\|u\|_1=\left(\int\int_{\Omega}\left(u^2+|\nabla u|^2\right)dx\,dy\right)^{1/2}$$

as $\partial u/\partial N$ on $\partial\Omega_2$ is a natural condition; it is a eliminated from the boundary conditions for F [here $H^1(\Omega)$ is the Sobolev space $\Rightarrow u\in C(\overline{\Omega})$].

- Since, here, f is a function that will render the calculations of the Ritz method more complicated [see (1.3)], it is recommendable that we return to the unknown function $U= U(x, y), (x, y)\in\Omega$. For this purpose, we put $U= u+v$ where $v= 1+\psi$ is a given function which verifies the boundary conditions of U (v is not subject to variation, it is a "variational constant": $\delta v = 0$). The energy functional becomes (the index $L_2(\Omega)$ is eliminated):

(2.1) $F(U) = (AU, U) - 2(f, U) + (Av, U) - (AU, v) + C_1(v)\,,\ U\in D(L)$

where $C_1(v)$ is a "variational constant" that can be eliminated ($\delta C_1(v) = 0$, v fixed). By using Green's formula, the following equalities can be inferred:

$$(Av,U)-(AU,v)=\int_{\partial\Omega_1}\frac{\partial u}{\partial N}ds+\int_{\partial\Omega_2}gU\,ds+C_2(v)$$

$$(AU,U)=\int\int_{\Omega}\nabla^T U\cdot Q\nabla U\,dx\,dy-\int_{\partial\Omega_1}\frac{\partial u}{\partial N}ds+\int_{\partial\Omega_2}gU\,ds$$

Subsequently, with the help of these equalities, we determine from (2.1) the energy functional [given below in (2.2)]. According to the theorem of the minimum of the energy functional, the variational problem equivalent (in generalized sense) with (1.1) is (U=1+u):

(Pv) Find the function $\tilde{u}\in\tilde{D}(F)$ so that $F(\tilde{u})=\min_{u\in\tilde{D}(F)}F(u)$ where F has the following expression

(2.2) $$F(u)=\frac{1}{2}\int\int_{\Omega}\left(\nabla^T u\cdot Q\nabla u-2fu\right)dx\,dy+\int_{\partial\Omega_2}gu\,dy\,,\ u\in\tilde{D}(F)$$

$$\tilde{D}(F)=\tilde{H}^1_{01}(\Omega)=\left\{u\in H^1_{01}(\Omega)\,\middle|\,\|u\|_A=\left(\int\int_{\Omega}\nabla^T u\cdot Q\nabla u\,dx\,dy\right)^{1/2}<\infty\right\}\equiv H_A$$

$$\left(\exists\tilde{c}_1,\tilde{c}_2>0,\tilde{c}_1\|u\|_1\le\|u\|_A\le\tilde{c}_2\|u\|_1\right)$$

where $\tilde{H}^1_{01}(\Omega)$ is the Sobolev space $H^1_{01}(\Omega)$ supplied with the energetic norm $\|\cdot\|_A$; this is the energetic space H_A (the completion of the linear space $D_0(F)$ in the energetic norm $\|\cdot\|_A$). We have $D_0(F)\subset H_A\subset H=L_2(\Omega)$ with $\|\cdot\|_H\le\|\cdot\|_{H^1}\le c\|\cdot\|_A$ [$D_0(F)$ is dense both in H_A and H; $H_A\subset H$ is a dense imbedding]. The generalized solution $\tilde{u}$ is determined approximately by:

b) *The Ritz Algorithm.* From $\tilde{D}(F)$, a linearly independent and complete in $\tilde{H}^1_{01}(\Omega)$ system of trial functions $\{\varphi_i\}$ is chosen [finite (non-orthogonal) basis in $D(F)$] and it is supposed that the n - order Ritz approximation for the exact solution u of the problem (Pv) is:

(2.3) $$u_n(x,y)=\sum_{k=1}^{n}c_k\varphi_k(x,y),\ c_k\ (\text{unknown})\in R^1\,;\ n=1,2,3,\ldots$$

that belongs to the set of functions $v_n=\sum_{1}^{n}a_k\varphi_k(x,y),\ \forall a_k\in R^1$; this set represents a linear subspace H_n so that the sequence $\{H_n\}=\{H_n|n\in\mathbb{N}^*\}$ is dense in the energetic space $H_A=\tilde{H}^1_{01}(\Omega)$. Then, the conditions $\frac{\partial F(u_n)}{\partial c_i}=0\,,\ i=\overline{1,n}$, are necessarily satisfied. These stationarity conditions of F on R^n at point (c_1, c_2, ..., c_n) are transformed into Ritz algebraic system:

(2.4) $$\sum_{j=1}^{n} K_{ij}c_j = b_i - g_i \; ; \left[or \; (u_n, \varphi_i)_A = (f_1 - g, \varphi_i)_{L_2} \right]; \; i = \overline{1,n}$$

where c_j, $j = \overline{1,n}$ are unknown and

$$K_{ij} \equiv (\varphi_i, \varphi_j)_A = \int\int_{\Omega} \nabla^T \varphi_i \cdot Q \nabla \varphi_j \, dx dy \, , \; b_i = \int\int_{\Omega} f_1 \varphi_i \, dx dy \, , \; g_i = \int_{\partial\Omega_2'} g\varphi dy$$

The notation is changed. The solution (2.3) and the system (2.4) are written in the form

(2.5) $$u_n(x,y) = \sum_{k=1}^{n}\sum_{m=1}^{n} c_{km} w_{km}(x,y) \, , \; c_{km} \in R^1$$

(2.6) $$\sum_{k,m=1}^{n} (w_{km}, w_{rs})_A c_{km} = (f_1, w_{rs})_{L(\Omega)} - (g_1, w_{rs})_{L(\partial\Omega_2')}$$

where the coefficients of the Ritz system have the values:

(2.7)
$$K_{kmrs} \equiv (w_{km}, w_{rs})_A = \int\int_{\Omega} \nabla^T w_{km} \cdot Q \nabla w_{rs} \, dx dy =$$
$$= \int\int_{\Omega} \left(\lambda_1 \frac{\partial w_{km}}{\partial x}\frac{\partial w_{rs}}{\partial x} + \lambda_2 \frac{\partial w_{km}}{\partial y}\frac{\partial w_{rs}}{\partial y} \right) dx dy \, ;$$
$$b_{rs} \equiv (f_1, w_{rs})_{L_2(\Omega)} = \int\int_{\Omega} f_1 w_{rs} \, dx dy \, ;$$
$$g_{rs} \equiv (g, w_{rs})_{L_2(\partial\Omega_2')} = \int_{\partial\Omega_2'} g w_{rs}(a,y) dy$$

c) *The choice of trial functions and the solving of the Ritz system.*

1. *Trigonometric polynomials.* A system of trial functions $w_{km} \in H^1_{01}(\Omega)$ $[w_{km} \in D_0(F)]$ of the following form is chosen:

(2.8) $$w_{km}(x,y) = \frac{\pi}{\sqrt{ab}} \left[\frac{a}{k\pi} \sin\frac{k\pi x}{a} - (x+a) \right] \left(\frac{b}{m\pi} \sin\frac{m\pi y}{b} + y \right); \; k,m = 1,3,5,...$$

These functions verify the boundary condition for $y = 0$ [$w_{km}(x, 0) = 0$ but according to the theory, verification of natural conditions is not obligatory (on $\overline{OC}$, $\overline{CB}$ and $\overline{AB}$)]. However, here $\partial w_{km}(0, y) / \partial x = 0$ (k, m = 1, 3, ...) but the condition on $\overline{AB}$ is not verified. The Ritz solution only verifies it approximately.

The properties of the trial function system. Let us consider the trigonometric functions

(2.9) $$\varphi_k(x) = \sqrt{\frac{2}{a}} \sin\frac{k\pi x}{a} \, , \; \psi_m(y) = \sqrt{\frac{2}{b}} \sin\frac{m\pi y}{b}$$

The function system $\{v_{km}\}$:

$$v_{km}(x,y) = \varphi_k(x)\psi_m(y), \; (x,y) \in \Omega \, ; \; k,m = 1,2,3,...$$

has the properties:

1°. $\{v_{km}\}$ belongs to the linear subspace $\widetilde{D}(F) \subset L_2(\Omega)$

2°. $\{v_{km}\}$ is orthonormal (i.e. linearly independent) in $L_2(\Omega)$:

$(v_{ij}, v_{km})_{L_2(\Omega)} = 0$ if $i \neq k$ or $i \neq m$ and $(v_{ij}, v_{km})_{L_2(\Omega)} = 1$ if $i = k$, $j = m$

3°. $\{v_{km}\}$ is complete in $L_2(\Omega)$: the Parseval equality holds, [2]:

(2.10) $$\sum_{k,m=1}^{\infty} \left| (u, v_{km})_{L_2(\Omega)} \right|^2 = \|u\|^2_{L_2(\Omega)} \, , \; \forall u \in C(\overline{\Omega}) \; - (\text{dense in } L_2(\Omega))$$

4°. $\{v_{km}\}$ is orthogonal in H^0_A (linearly independent) but it is not orthonormal in $H^0_A(\Omega)$. Here $H^0_A [\equiv H^0_A(\Omega)]$ is the space $D_0(F)$ supplied with the energetic product $(.,.)_A$ and energetic norm defined by the following equalities (A is positive definite)

(2.11)
$$(u,v)_A = \int\int_{\Omega} \nabla^T u \cdot Q \nabla v \, dx dy ; \; u,v \in D_0(F)$$
$$\|u\|_A = \sqrt{(u,u)_A} \quad ; \; u, \in D_0(F)$$

Indeed, simple calculations lead to the equalities and the orthonormalized system $\{\overline{v}_{km}\}$

$$(v_{km}, v_{rs})_A = \begin{cases} 0 & , \text{ if } k \neq r, \; m \neq s \\ \dfrac{\pi^2}{a^2 b^2} \left(\lambda_1 k^2 b^2 + \lambda_2 m^2 a^2 \right) \left(= \|v_{km}\|^2_A \right) & , \; if \; k = r, \; m = s \end{cases}$$

$$\overline{v}_{km} = \frac{v_{km}}{\|v_{km}\|_A}$$

5°. The system $\{v_{km}\}$ is also complete with respect to the energetic norm in the energetic Hilbert space H_A - the completion of the subspace H_A^0 with respect to the convergence in energy. Indeed, according to the theorem : A - positive definite on $D(A)$, $\varphi_k \in D(A)$ - dense in H_A and $\{A\varphi_k\}$ - complete in H implies $\{\varphi_k\}$ - complete in H_A, a simple calculus shows that

$$Av_{km} = -\nabla\cdot(Q\nabla v_{km}) = \pi\left(\lambda_1 k^2 a^{-2} + \lambda_2 m^2 b^{-2}\right)v_{km}$$

Consequently, since $\{v_{km}\}$ is complete in $L_2(\Omega)$, $\{Av_{km}\}$ is complete in $H = L_2(\Omega)$ [multiplication with a constant ($\neq 0$) maintains completion] and then, $\{v_{km}\}$ is complete in energetic norm (in H_A).

Remark. Taking into account the system $\{\tilde\varphi_k\}$ in which $\tilde\varphi_k(x,y) = \sqrt{\dfrac{2}{a}}\, y \sin\dfrac{k\pi x}{a}$,

we have

$$(\tilde\varphi_k, \tilde\varphi_m)_A = \begin{cases} 0 & \text{, if } k \neq m \\ \dfrac{1}{3} b^3 \lambda_1 \left(\dfrac{k\pi}{a}\right)^2 + \lambda_2 \quad \left(= \|\tilde\varphi_k\|_A^2\right) & \text{; if } k = m \end{cases}$$

and hence $\{\tilde\varphi_k\}_1^\infty$ is orthogonal in H_A^0.

Now, using (2.8), the scalar products (2.7) have the expressions:

(2.12)
$$K_{kmrs} = (w_{km}, w_{rs})_A =$$
$$= \begin{cases} \dfrac{3}{4}\left[5\lambda_1 b^2 \dfrac{1}{m^2} - 11\lambda_2 a^2 \dfrac{1}{k^2} + \dfrac{2\pi^2}{3}\left(\lambda_1 b^2 + 7\lambda_2 a^2\right)\right] & ;\ k = r, m = s \\ \lambda_1 b^2 \dfrac{m^2+s^2}{m^2 s^2} - 3\lambda_2 a^2 \dfrac{k^2+r^2}{k^2 r^2} + \dfrac{\pi^2}{3}\left(\lambda_1 b^2 + 7\lambda_2 a^2\right) & ;\ k \neq r, m \neq s \\ \dfrac{3}{2}\lambda_1 b^2 \dfrac{m^2+s^2}{m^2 s^2} - \dfrac{11}{2}\lambda_2 a^2 \dfrac{1}{k^2} + \dfrac{\pi^2}{3}\left(\dfrac{3}{2}\lambda_1 b^2 + 7\lambda_2 a^2\right) & ;\ k = r, m \neq s \\ \dfrac{5}{2}\lambda_1 b^2 \dfrac{1}{m^2} - \dfrac{9}{2}\lambda_2 a^2 \dfrac{k^2+r^2}{k^2 r^2} + \dfrac{\pi^2}{3}\left(\lambda_1 b^2 + \dfrac{21}{2}\lambda_2 a^2\right) & ;\ k \neq r, m = s \end{cases}$$

$$(f_1, w_{rs})_{L_2(\Omega)} = \int_0^a\int_0^b f_1(x,y)\, w_{rs}(x,y)\, dx\, dy$$

$$(g, w_{rs})_{L_2(\partial\Omega_2'')} = \int_0^b g\, w_{rs}(a,y)\, dy = -\pi g b\sqrt{ab}\left[\left(\frac{2}{s\pi}\right)^2 + 1\right]$$

These values are introduced into the Ritz system (2.6) which becomes a Cramer system with a unique solution $c_{km} = c^*_{km}$ $(k, m = 1,3,5,\ldots)$; the matrix of (2.6) is nonsingular, symmetrical and positive definite like the operator A.

Numerical application (n=3). We choose the values a=2, b=1, f_1=0, $\lambda_1=\lambda_2=1$, $g=-1$, [$\partial u/\partial x = 1$ on $\overline{AB}$ indicates a heat flux on $\overline{AB}$ towards the inside of the plate]. The Ritz solution has the form

(2.13) $$u_3(x,y) = \sum_{k,m=1}^{3} c_{km} w_{km}(x,y) = c_{11}w_{11} + c_{31}w_{31} + c_{13}w_{13} + c_{33}w_{33}$$

The coefficients of the Ritz system are ($\pi = 3.14$)

$$(w_{11}, w_{11})_A = 113.8592638\ ;\quad (w_{13}, w_{14})_A = 76.71777661$$
$$(w_{31}, w_{11})_A = 123.9643297\ ;\quad (w_{33}, w_{11})_A = 83.18395365$$
$$(w_{13}, w_{13})_A = 110.5259305\ ;\quad (w_{31}, w_{13})_A = (w_{33}, w_{11})_A$$
$$(w_{33}, w_{13})_A = 121.7421075\ ;\quad (w_{31}, w_{31})_A = 143.1927971$$
$$(w_{33}, w_{31})_A = 96.27333217\ ;\quad (w_{33}, w_{33})_A = 139.8592638$$
$$(-1, w_{11}) = (-1, w_{31}) = 6.24351557\ ;\quad (-1, w_{13}) = (-1, w_{33}) = 4.642953231$$

By means of the Gauss method, the following solution has been found for the Ritz system

(2.14) $$c_{11} = -0.125130\ ;\ c_{13} = -0.004312$$
$$c_{31} = -0.068856\ ;\ c_{33} = -0.002417$$

with the residual error of the solution c_{km} of the 4.10^{-9} (only six exact decimals have been considered in the (2.14)).

The Ritz solution in the 3 rd order approximation is

(2.15)
$$u_3(x,y) = \frac{\pi}{\sqrt{2}}\left[c_{11}\left(\frac{2}{\pi}\sin\frac{\pi x}{2} - x - 2\right)\left(\frac{1}{\pi}\sin\pi y + y\right) + \right.$$
$$+ c_{13}\left(\frac{2}{\pi}\sin\frac{\pi x}{2} - x - 2\right)\left(\frac{1}{3\pi}\sin 3\pi y + y\right) +$$
$$+ c_{31}\left(\frac{2}{3\pi}\sin\frac{3\pi x}{2} - x - 2\right)\left(\frac{1}{\pi}\sin\pi y + y\right) +$$
$$\left. + c_{33}\left(\frac{2}{3\pi}\sin\frac{3\pi x}{2} - x - 2\right)\left(\frac{1}{3\pi}\sin 3\pi y + y\right)\right]$$

For example, we obtain the values ($U_3(x, y) = 1 + u_3(x, y)$):

$u_3(2;\ 1/2)=0.43274$; $u_3(1;\ 1/2)=0.15123$; $u_3(0;\ 1/2)=0.2163$; $u_3(1,\ 1)=0.20549$

The polynomial type trial functions. Chebyshev polynomials. The Chebyshev polynomials $T_i(x)$, $x\in R^1$ are determined by recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, 3, \ldots$$

$$\left(\text{with } T_0(x)=1,\ T_1(x)=x;\ (u,v)_{2,\rho}=\int_{-1}^{1}\rho(x)uv\,dx, \rho(x)=\frac{1}{\sqrt{1-x^2}}\right)$$

These polynomials can be introduced by means of the Schmidt orthogonalization of the linear independent system $\left\{x^n\right\}_{n=0}^{\infty}$ with respect to the scalar product $L_{2,\rho(-1,1)}$. In the space $L_{2,\rho(-1,1)}$ the system $\overline{T}_n=\sqrt{2/\pi}\,T_n(x)$ is orthonormal [and complete: $\forall u\in C^2(-1,1),\ (u,T_i)_{2,\rho}=0\Rightarrow u(x)\equiv 0$

In the two-dimensional case the Chebyshev polynomials $p_k(x, y)$ can be introduced by means of the formulas

(2.16) $$p_k(x, y) = T_i(x)T_j(y),\ k = 1, 2, 3, \ldots$$

with $k=\frac{1}{2}(i+j)(i+j+1)+j+1$; $i, j = 0,1,2,\ldots$

The first polynomials are:

$$p_1(x, y) = T_0(x)T_0(y) = 1,\ p_2(x, y) = x,\ p_3(x, y) = y,\ p_4(x, y) = 2x^2-1$$

$$p_5(x, y) = xy,\ p_6(x, y) = 2y^2 - 1, \ldots, p_{10}(x, y) = 4y^3 - 3y, \ldots$$

We consider the arbitrary $u\in C^2(\Omega)$ on the domain $\Omega = (-1, 1) \times (-1, 1)$ of the plane Oxy. We notice that the implication holds:

$$0=\left(u,p_k(x,y)\right)_{2,\rho}=\int_{-1}^{1}T_j(y)\left(\int_{-1}^{1}\rho u T_i\,dx\right)dy=\int_{-1}^{1}T_j\left(u,T_i(x)\right)_{2,\rho}dy\Rightarrow$$

$$\Rightarrow(u,T_i)_{2,\rho}=0\Rightarrow u(x,y)\equiv 0,\ (x,y)\in\Omega$$

Therefore $\{p_k\}$ is a complete orthogonal system on Ω.

The fourth order Ritz approximation (n=4; a=2, b=1, λ_1=λ_2=1, f_1=0, g=-1; $\partial u/\partial x$=1 on x=2). The Ritz solution is chosen, for the variational problem (Pv), in the form

$$u_4(x,y)=\sum_{k=1}^{4}c_k\varphi_k(x,y),\ \varphi_k(x,y)=yp_k(x,y),\ k=\overline{1,4}$$

The coefficients K_{ij}, b_i given in formulas (2.7) are calculated exactly and have the values:

$$K_{11}=2\ ;\ K_{12}=2\ ;\ K_{13}=2\ ;\ K_{14}=\frac{10}{3}\ ;\ K_{22}=\frac{10}{3}\ ;$$

$$K_{23}=2\ ;\ K_{24}=\frac{26}{3}\ ;\ K_{33}=\frac{8}{3}\ ;\ K_{34}=\frac{10}{3}\ ;\ K_{44}=\frac{1402}{45}$$

$$g_1=\frac{1}{2}\ ,\ g_2=1\ ,\ g_3=\frac{1}{3}\ ,\ g_4=\frac{7}{4}$$

We solve the Ritz system (2.4) by means of the Gauss method:

$$\left[\begin{array}{c:ccc|c} 2 & 2 & 2 & 10/3 & 1/2 \\ \hdashline 2 & 10/3 & 2 & 26/3 & 1 \\ 2 & 2 & 8/3 & 10/3 & 1/3 \\ 10/3 & 26/3 & 10/3 & \frac{1402}{45} & 7/2 \end{array}\right] \xrightarrow{(1)} \left[\begin{array}{c:cc|c} 8/3 & 0 & 32/3 & 1 \\ \hdashline 0 & 4/3 & 0 & -1/3 \\ 32/3 & 0 & \frac{256}{5} & 16/3 \end{array}\right] \xrightarrow{(2)}$$

$$\xrightarrow{(2)} \left[\begin{array}{c:c|c} 32/9 & 0 & -8/9 \\ \hdashline 0 & \frac{1024}{45} & 32/9 \end{array}\right] \xrightarrow{(3)} \left(\frac{32}{9}\right)^2 \left[\begin{array}{c|c} \frac{32}{5} & 1 \end{array}\right]$$

The Ritz system has the solution

(2.18) $$c_4=\frac{5}{32}=0.15625\ ;\ c_3=c_2=-\frac{1}{4}=-0.25000\ ;\ c_1=0.48958$$

We obtain the values:

(2.19) $$u_4(2;1/2)=0.4791;\ u_4(1;1/2)=0.13542;$$
$$u_4(0;1/2)=0.1041;\ u_4(1,1)=0.14583$$

Solving on the computer. A TURBO-PASCAL computer program has been used in order to perform tha calculations of the Ritz algorithm (The program is presented in [8]). The program is applied to the approximation of the boundary value problem (1.1) by means of the Chebyshev polynomials. The numerical results (the values of the solution $c_k^{(n)}$ of the Ritz system) are given up to the $n = 28$ approximation (see Table 1). Only four decimals are considered for the values. We notice that in the case $n = 4$ the values in Table 1 coincide with those calculated (without programming) and given in (2.18)

By using Table 1 the following values for $u_n(1, 1/2)$ are obtained (Table 2).

Table 1

The solutions $c_k = c_k^{(n)}$

k/n	3	5	6	10	15	20	25	28
1	0.1250	0.4896	0.1496	0.1370	0.1824	0.0136	-0.7197	0.0146
2	0.3750	-0.2499	0.0900	0.0071	-0.1734	0.0226	1.2913	-0.0452
3	-0.2500	-0.2500	0.1386	-0.1062	-0.0102	0.1782	1.5583	0.3174
4		0.1563	0.1563	0.1089	0.0880	-0.1062	-0.4244	-0.0506
5			-0.3886	0.1971	0.2111	0.0483	2.3159	-0.0522
6				-0.0837	0.0677	-0.1734	-0.9311	-0.1989
7				0.0349	0.0473	0.0374	-0.0938	-0.1009
8				-0.2045	-0.2561	0.2401	0.9188	0.2869
9				0.0837	0.0299	0.0555	1.3163	-0.0309
10				0.0000	0.0068	-0.0739	0.4780	0.2041
11					-0.0085	0.0259	0.0382	0.0391
12					0.1130	-0.0618	0.1212	0.1314
13					-0.0299	-0.2061	-0.5339	-0.1582
14					0.0000	0.1754	-0.7432	-0.2282
15						-0.0080	-0.0076	0.0392
16						0.0008	0.0096	0.0096
17						-0.0356	-0.0609	-0.0622
18						0.0923	-0.0785	-0.0822
19						-0.0609	0.3740	0.2294
20						0.0080	0.0076	-0.0874
21							0.0000	0.0035
22							0.0001	0.0000
23							-0.0131	-0.0131
24							0.0563	0.0568
25							-0.0716	-0.0717
26								0.0290
27								-0.0035
28								0.0000

Table 2

n	3	4	5	10	15	20	25	28
$U_n\left(1,\frac{1}{2}\right)$	1.1875	0.1355	0.1355	0.1156	0.1203	0.1270	0.1269	0.1270

c) *The error in the energetic norm of the Ritz approximate solution* $u_n(\in D(A))$.

We put $Au_n = f_n$ and $u_n = \tilde{u}_n$, where $\tilde{u}_n$ is the generalized solution for $Au = f_n$.

If $\tilde{u}(\in H_A)$ is the generalized solution for $Au = f$ (A-positive definite; H_A-energetic space) and the Riesz theorem is used on H_A (H_A-Hilbert space) we have $(u \in H_A)$

(2.20) $\quad A\tilde{u}_n = f_n \Rightarrow (\tilde{u}_n, u)_A = (f_n, u)_{L_2}\ ;\ (\tilde{u}, u) = (f, u)_{L_2}\ ;\ \|u\|_{L_2} \le \frac{1}{\alpha}\|u\|_A$

If we use (2.20), the following relations can be written

(a) $\quad (\tilde{u}_n - \tilde{u}, u)_A = (f_n - f, u)_{L_2}\ ,\ \forall u \in H_A\ ;\ (b)\ \|u\|_A \le \frac{1}{\alpha}\|f\|_{L_2}$

From (a) it results that $\tilde{v} = \tilde{u}_n - \tilde{u}$ is a generalized solution for the equation $A_v = f_n - f$ and, if (b) is further considered, we get the estimation

(2.21) $$\|\tilde{u}_n - \tilde{u}\|_A \le \frac{1}{\alpha}\|Au_n - f\|_{L_2}$$

If $\tilde{u} \in D(A)$ [then $\tilde{u} = u_0$ is the classical solution: $Au_0 = f$] and if $f = 0$, from (2.21), we obtain the estimation (the error)

(2.22) $$\|u_n - u_0\|_A \le \frac{1}{\alpha}\|Au_n\|_{L_2}\ ,\ \alpha = \frac{1}{\sqrt{C_F}}\ ,\ u_n \in D_0(F)$$

where C_F is Friederichs' constant (1.8).

Remark. It is known that for an orthonormalized trial function system $\{\varphi_k\}(\subset D(A))$ we have $\|Au_n - f\|_{L_2} \to 0$, as $n \to \infty$ and that the Ritz algorithm is stable [2], [1].

Approximation by trigonometric trial functions. For $n = 3$ $(u_3 \in D_0(F))$:

$$\|Au_3\|_{L_2}^2 = \int\int_{\Omega}\left[\nabla u_3(x,y)\right]^2 dxdy = \sum_{k,m=1}^{3}\sum_{r,s=1}^{3} c_{km}c_{rs}I_{kmrs}$$

where $(k, m, r, s \neq 2)$

$$I_{kmrs} = \int_0^2\int_0^1 \nabla w_{km}\nabla w_{rs}\,dxdy =$$

$$= \pi^2\begin{cases} \dfrac{5k^2}{16m^2} - \dfrac{11m^2}{k^2} - \dfrac{15}{2} + \pi^2\left(\dfrac{k^2}{24} + \dfrac{14m}{3}\right) & , k = r,\ m = s \\ k^2\left(\dfrac{1}{8m^2} + \dfrac{1}{8s^2} - \dfrac{5}{k^2} + \dfrac{1}{24}\pi^2\right) & , k = r,\ m \neq s \\ -m^2\left(\dfrac{2}{k^2} + \dfrac{2}{r^2} + \dfrac{7}{m^2} - \dfrac{14}{3}\pi^2\right) & , k \neq r,\ m = s \\ -12 & , k \neq r,\ m \neq s \end{cases}$$

Then we have

(2.23) $$\|Au_3\|_{L_2}^2 \approx 0.2691 \text{ and } \|u_3 - u_0\|_A \le 0.519\sqrt{C_F}$$

Approximation by Chebyshev polynomials. In this case, we have

$$u_n \in D_0(F),\ \|Au_n\|_{L_2}^2 = \iint_\Omega \left[\sum_{k=1}^n c_k\left(y\nabla p_k + 2\frac{\partial p_k}{\partial y}\right)\right]^2 \mathrm{d}x\mathrm{d}y$$

For $n = 4$ we obtain

(2.24) $$\|Au_4\|_{L_2}^2 \approx 0.135417 \text{ and } \|u_4 - u_0\|_A \le 0.368\sqrt{C_F}$$

Remark. With the Chebyshev polynomials the error is smaller than the resulting from the employment of the trigonometric functions.

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