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ON SOME STEFFENSEN-TYPE ITERATIVE METHODS FOR A CLASS OF NONLINEAR EQUATIONS

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INTRODUCTION

Consider a Banach space X and the equation

(1) $$F(x) + G(x) = 0,$$

where $F, G: X \to X$ are nonlinear operators, F being Fréchet differentiable and G being continuously but non differentiable. This is the case when we study an equation $H(x) = 0$, with $H: X \to X$ a non differentiable operator to which we can't apply Newton's method. H is then split into two parts: a differentiable part and a non differentiable one.

Various methods have been proposed for solving these kind of problems:

In [8,9,10] are considered the Newton-like methods:

(2) $$x_{n+1} = x_n - F'(x_n)^{-1}(F(x_n) + G(x_n)), \quad x_0 \in X, \quad n = 0, 1,\ldots$$

and, more generally,

(3) $$x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)), \quad x_0 \in X, \quad n = 0, 1,\ldots$$

where A is a linear operator approximating F'.

In [1] is studied the secant-type method

(4) $$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1}(F(x_n) + G(x_n)), \quad x_0, \quad x_1 \in X, \quad n = 1, 2,\ldots,$$

$[x,y;F]$ denoting the first order divided difference of F on the points x, y.

The convergence order of these sequences is linear, as it can also be seen in the numerical example.

In [3] is considered a combination of Newton's method and the secant method:

(5) $$x_{n+1} = x_n - (F'(x_n) + [x_{n-1}, x_n; G])^{-1}(F(x_n) + G(x_n)), \quad x_0, \quad x_1 \in X, \quad n = 1, 2,\ldots$$

having the convergence order $\frac{1+\sqrt{5}}{2} \approx 1618$ i.e. the convergence order of the secant method.

In the present paper we propose a method based on Steffensen's method and Newton's method, having quadratic convergence:

(6) $x_{n+1} = x_n - (F'(x_n) + [x_n, \varphi(x_n); G])^{-1}(F(x_n) + G(x_n)), \quad x_0 \in X, n = 0, 1, \ldots,$

where $\varphi: X \to X, \varphi(x) = x - \lambda(F(x) + G(x)), \lambda$ being a fixed positive number.

2. THE CONVERGENCE OF THE METHOD

We shall use, as in [4,5] the known definitions for the divided differences of an operator:

DEFINITION 1. *An operator $[x_0, y_0; G]$ belonging to the space $\mathscr{L}(X, X)$ (the Banach space of the linear and bounded operators from X to X) is called the first order divided difference of the operator $G: X \to X$ on the points $x_0, y_0 \in X$ if the following properties hold:*

a) $[x_0, y_0; G](y_0 - x_0) = G(y_0) - G(x_0)$, *for* $x_0 \neq y_0$;

b) *if G is Fréchet differentiable at x_0, then* $[x_0, x_0; G] = G'(x_0)$.

DEFINITION 2. *An operator belonging to the space $\mathscr{L}(X, \mathscr{L}(X, X))$, denoted by $[x_0, y_0, z_0; G]$, is called the second-order divided difference of the operator $G: X \to X$ on the points $x_0, y_0, z_0 \in X$ if the following properties hold:*

a) $[x_0, y_0, z_0; G](z_0 - x_0) = [y_0, z_0; G] - [x_0, y_0; G]$, *for the distinct points* $x_0, y_0, z_0 \in X$;

b) *if G is two times Fréchet differentiable at $x_0 \in X$, then* $[x_0, x_0, x_0; G] = \frac{1}{2} G''(x_0)$.

We shall denote by $B_r(x_0) = \{x \in X \mid \|x - x_0\| < r\}$ the open ball having the center at $x_0 \in X$ and the radius $r > 0$.

Concerning the convergence of the iterative process (6):

$$x_{n+1} = x_n - \left(F'(x_n) + [x_n, \varphi(x_n); G]\right)^{-1}\left(F(x_n) + G(x_n)\right), \quad x_0 \in X, \quad n = 0, 1, \ldots$$

where $\varphi: X \to X$, $\varphi(x) = x - \lambda(F(x) + G(x))$, we'll prove the following theorem:

THEOREM 3. *If there exists the element $x_0 \in X$, and the positive real numbers K, l, ε, M, r such that:*

i) *G is continuous on $B_r(x_0)$;*

ii) *F is Fréchet differentiable on $B_r(x_0)$, with the Fréchet derivative satisfying the Lipschitz condition*

$$\|F'(x) - F'(y)\| \leq l\|x - y\|, \forall x, y \in B_r(x_0);$$

iii) *The second-order divided difference of G is uniformly bounded on $B_r(x_0)$:*

$$\|[x, y, z; G]\| \leq K, \forall x, y, z \in B_r(x_0);$$

iv) *The operators $F'(x) + [x, \varphi(x); G]$ are invertible, with the inverses uniformly bounded:*

$\forall x \in B_r(x_0)$ *with* $\varphi(x) \in B_r(x_0)$ *there exists* $\left(F'(x) + [x, \varphi(x); G]\right)^{-1}$ *and*

$$\left\|\left(F'(x) + [x, \varphi(x); G]\right)^{-1}\right\| \leq M;$$

v) *λ is chosen such that $\lambda \leq M$;*

vi) $q := M^2 \varepsilon \left(\frac{l}{2} + 2K\right) < 1$ *and the radius is given by*

$$r := \frac{1}{M\left(\frac{l}{2} + 2K\right)} \sum_{k=0}^{\infty} q^{2^k},$$

then:

j) *The sequence $(x_n)_{n \geq 0}$ given by (6) is well defined and $(x_n) \subset B_r(x_0)$;*

jj) *$(x_n)_{n \geq 0}$ converges to some $x^* \in \overline{B_r(x_0)}$, which is a solution of equation (1);*

jjj) *We have the estimation*

$$\|x^* - x_n\| \leq \frac{q^{2^n}}{M\left(\frac{l}{2} + 2K\right)\left(1 - q^{2^n}\right)}$$

Proof. From the hypothesis i) concerning F it is known [6] that we get

(7) $$\|F(y) - F(x) - F'(x)(y - x)\| \leq \frac{l}{2}\|y - x\|^2.$$

From the definitions of the divided differences we obtain

(8) $$G(y) - G(x) - [x, \varphi(x); G](y - x) = [x, \varphi(x), y; G](y - \varphi(x))(y - x).$$

Indeed, $[x,\varphi(x),y;G](y-\varphi(x))(y-x) = [\varphi(x),y;G](y-\varphi(x)) - [x,\varphi(x);G](y-\varphi(x)) = G(y)-G(\varphi(x)) + [x,\varphi(x);G](\varphi(x)-x) - [x,\varphi(x);G](y-x) =$
$= G(y)-G(\varphi(x)) + G(\varphi(x)) - G(x) - [x,\varphi(x);G](y-x) =$
$= G(y)-G(x)-[x,\varphi(x);G](y-x).$

We shall prove by induction that

(9)
$$x_k,\varphi(x_k)\in B_r(x_0),\ k\in \mathbf{N}$$
$$\|F(x_k)+G(x_k)\|\le M^{-2}\left(\frac{l}{2}+2K\right)^{-1} q^{2^k},\ k\in \mathbf{N}.$$

From the above inequality it can be easily deduced by (6) that $\exists x_{k+1}$ and

(10)
$$\|x_{k+1}-x_k\| = \left\|\left(F'(x_k)+[x_k,\varphi(x_k);G]\right)^{-1}\left(F(x_k)+G(x_k)\right)\right\| \le$$
$$\le M^{-1}\left(\frac{l}{2}+2K\right)^{-1} q^{2^k}$$

For $k=0$ we have:

$x_0\in B_r(x_0)$;

$\|x_0-\varphi(x_0)\| = \|x_0-x_0+\lambda(F(x_0)+G(x_0))\| \le \lambda\varepsilon \le M\varepsilon < r$, which imply that $\varphi(x_0)\in B_r(x_0)$;

$$\|F(x_0)+G(x_0)\|\le\varepsilon = M^{-2}\cdot\left(\frac{l}{2}+2K\right)^{-1}\cdot q^{2^0}.$$

Suppose now that (9) is true for $k=\overline{1,n-1}$. By (10) it follows that $\exists x_n$, and we have that $x_n\in B_r(x_0)$. Indeed, $\|x_n-x_0\|\le\|x_1-x_0\|+\ldots+\|x_n-x_{n-1}\|\le$
$$\le M^{-1}\left(\frac{l}{2}+2K\right)^{-1}\sum_{k=0}^{n-1} q^{2^k} < r.$$

Then, using (6), (7), (8) and (9),

$$\|F(x_n)+G(x_n)\| \le \|F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n-x_{n-1})\| +$$
$$+\|G(x_n)-G(x_{n-1})-[x_{n-1},\varphi(x_{n-1});G](x_n-x_{n-1})\| \le$$
$$\le \frac{l}{2}\|x_n-x_{n-1}\|^2 + K\|x_n-x_{n-1}\|\cdot\|x_n-\varphi(x_{n-1})\| \le \frac{l}{2}M^{-2}\left(\frac{l}{2}+2K\right)^{-2} q^{2^n} +$$

$$+KM^{-1}\left(\frac{l}{2}+2K\right)^{-1} q^{2^{n-1}}\left(M^{-1}\left(\frac{l}{2}+2K\right)^{-1} q^{2^{n-1}} + \lambda M^{-2}\left(\frac{l}{2}+2K\right)^{-1} q^{2^{n-1}}\right) \le$$
$$\le M^{-2}\left(\frac{l}{2}+2K\right)^{-1} q^{2^n}.$$

It remains to show that $\varphi(x_n)\in B_r(x_0)$:

$$\|x_0-\varphi(x_n)\| \le \|x_0-x_n\| + \lambda\|F(x_n)+G(x_n)\| \le$$
$$\le M^{-1}\left(\frac{l}{2}+2K\right)^{-1}\sum_{k=0}^{n-1} q^{2^k} + M^{-1}\left(\frac{l}{2}+2K\right)^{-1} q^{2^n} < r.$$

The induction (9) is proved.

Now we prove that the sequence $(x_n)_{n\ge0}$ is a Cauchy sequence, hence it converges to some element $x^*\in B_r(x_0)$:

$$\|x_{n+p}-x_n\| \le \|x_{n+p}-x_{n+p-1}\| + \ldots + \|x_{n+1}-x_n\| \le M^{-1}\left(\frac{l}{2}+2K\right)^{-1}\sum_{k=n}^{n+p-1} q^{2^k} =$$
$$= M^{-1}\left(\frac{l}{2}+2K\right)^{-1} q^{2^n}\sum_{k=n}^{n+p-1} q^{2^k-2^n} \le M^{-1}\left(\frac{l}{2}+2K\right)^{-1} q^{2^n}\left(1+q^{2\cdot2^n-2^n}+q^{4\cdot2^n-2^n}+\ldots\right) <$$
$$< M^{-1}\left(\frac{l}{2}+2K\right)^{-1}\frac{q^{2^n}}{1-q^{2^n}}.$$

Passing to limit for $n\to\infty$ in relation (6) and taking into account the hypotheses concerning F and G, we get that x^* is a solution of (1).

The estimation jjj) is obtained from the above relation, for $p\to\infty$. □

3. NUMERICAL EXAMPLES

Given the system

$$\begin{cases} 3x^2y+y^2-1+|x-1|=0 \\ x^4+xy^3-1+|y|=0 \end{cases}$$

we'll consider $X=\left(\mathbf{R}^2,\|\cdot\|_\infty\right)$ and $F,G:X\to X$, $F=(f_1,f_2)$, $G=(g_1,g_2)$, with $f_1(x,y)=3x^2y+y^2-1$, $f_2(x,y)=x^4+xy^3-1$, $g_1(x,y)=|x-1|$, $g_2(x,y)=|y|$.

We shall take $[x,y;G]\in M_2(\mathbf{R})$ given by

$$[x,y;G](i,1)=\frac{g_i(y^1,y^2)-g_i(x^1,y^2)}{y^1-x^1},\ \text{and}\ [x,y;G](i,2)=\frac{g_i(x^1,y^2)-g_i(x^1,x^2)}{y^2-x^2},$$

$i=1,2.$

Using the method (2) with $x_0=(1,0)$, we obtain

n	x_n^1	x_n^2	$\lVert x_n-x_{n-1}\rVert$
0	1.00000000000000E+0000	0.00000000000000E+0000	
1	1.00000000000000E+0000	3.33333333333333E-0001	3.33E-0001
2	9.06550218340611E-0001	3.54002911208151E-0001	9.344E-0002
3	8.85328400663412E-0001	3.38027276361332E-0001	2.122E-0002
4	8.91329556832800E-0001	3.26613976593566E-0001	1.141E-0002
...			
39	8.94655373334687E-0001	3.27826521746298E-0001	5.149E-0019

Using the method (5) with $x_0=(1,1)$, $x_1=(2,2)$ we get:

n	x_n^1	x_n^2	$\lVert x_n-x_{n-1}\rVert$
0	2.00000000000000E+0000	2.00000000000000E+0000	
1	1.00000000000000E+0000	1.00000000000000E+0000	1.000E+0000
2	3.33333333333333E-0001	1.33333333333333E+0000	6.666E+0001
3	9.62025316455696E-0001	3.54430379746835E-0001	9.789E-0001
4	9.00696217156264E-0001	3.30465935597986E-0001	6.132E-0002
5	8.94706409693425E-0001	3.27855252188766E-0001	5.989E-0003
6	8.94655376809408E-0001	3.27826524565125E-0001	5.103E-0005
7	8.94655373334687E-0001	3.27826521746298E-0001	3.474E-0009
8	8.94655373334687E-0001	3.27826521746298E-0001	2.003E-0017
9	8.94655373334687E-0001	3.27826521746298E-0001	2.710E-0020

Using method (6) with $\lambda=0.5$ and $x_0=(1,1)$ we get:

n	x_n^1	x_n^2	$\lVert x_n-x_{n-1}\rVert$
0	1.00000000000000E+0000	0.00000000000000E+0000	
1	1.40000000000000E+0000	0.00000000000000E+0000	1.000E+0000
2	1.15421294962624E+0000	1.43841335097579E-0001	2.457E-0001
3	1.01057150046324E+0001	2.69169893550861E-0001	1.436E-0001
4	8.99073392876452E-0001	3.76267383109311E-0001	1.114E-0001
5	8.95022505657807E-0001	3.28815382034089E-0001	4.745E-0002

6	8.94655504144107E-0001	3.27827488746546E-0001	9.878E-0004
7	8.94655373334787E-0001	3.27826521746806E-0001	9.669E-0007
8	8.94655373334687E-0001	3.27826521746298E-0001	5.086E-0013
9	8.94655373334687E-0001	3.27826521746298E-0001	2.710E-0020

It seems that the best results are not obtained here for λ taken too small, because the divided differences can't be computed in this case for $\lVert x_n-x_{n-1}\rVert\le 1.E-16$.

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