REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 24, N^{os} 1–2, 1995, pp. 45–52

SELECTIONS ASSOCIATED TO THE METRIC PROJECTION

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Let X be a normed space, M a subspace of X and x an element of X. The *distance* from x to M is defined by

(1) $$d(x, M) := \inf\{\|x - y\| : y \in M\}.$$

An element $y \in M$ verifying the equality $\|x - y\| = d(x, M)$ is called an *element of best approximation* for x by elements in M. The set of all elements of best approximation for x is denoted by $P_M(x)$, i.e.

(2) $$P_M(x) := \{y \in M : \|x - y\| = d(x, M)\}.$$

If $P_M(x) \neq \varnothing$ (respectively $P_M(x)$ is a singleton) for all $x \in X$, then M is called a *proximinal* (respectively a *Chebyshevian*) subspace of X.

The set-valued application $P_M : X \to 2^M$ is called the *metric projection* of X on M and a function $p : X \to M$ such that $p(x) \in P_M(x)$, for all $x \in X$, is called a *selection* for the metric projection P_M. Observe that the existence of a selection for P_M implies $P_M(x) \neq \varnothing$, for all $x \in X$, i.e. the subspace M is necessarily proximinal.

The set

(3) $$\operatorname{Ker} P_M := \{x \in X : 0 \in P_M(x)\},$$

is called the *kernel* of the metric projection P_M.

In many situations for a given subspace M of X the problem of best approximation is not considered for the whole space X but rather for a subset K of X. This is the case which we consider in this paper and to this end we need some definitions and notation.

If K is a subset of the normed space X and $P_M(x) \neq \varnothing$ (respectively $P_M(x)$ is a singleton) for all $x \in K$, then the subspace M is called *K-proximinal* (respectively *K-Chebyshevian*). The restriction of the metric projection P_M to K is denoted by $P_{M|K}$ and its kernel by Ker $P_{M|K}$.

(4) $$\operatorname{Ker} P_{M|K} := \{x \in K : O \in P_M(x)\}.$$

For two nonvoid subsets U, V of X denote by $U + V := \{u + v : u \in U, v \in V\}$ their algebraic sum. If every $x \in U + V$ can be uniquely written in the form $x = u + v$ with $u \in U$ and $v \in V$, then $U + V$ is called the *direct algebraic sum* of the sets U and V and is denoted by $U \dot{+} V$. If $K = U \dot{+} V$ and the application $(u, v) \to u + v$, $u \in U$, $v \in V$, *is a topological homeomorphism between* $U \times V$ (endowed with the product topology) and K then K is called the *direct topological sum* of the sets U and V, denoted by $K = U \oplus V$.

F. Deutsch [2] proved that if M is a proximinal subspace of X then the metric projection P_M admits a continuous and linear selection if and only if the subspace M is complemented in X by a closed subspace of $\operatorname{Ker} P_M$ ([2], Theorem 2.2).

In [4], one of the authors of the present paper considered a similar problem for a closed convex cone K in X and a K-proximinal subspace M of X, asking for $P_{M|K}$ to admit a continuous, positively homogeneous and additive selection. The following sufficient condition for the existence of such a selection was obtained:

If there exist two closed convex cones $C \subset \operatorname{Ker} P_M$ and $U \subset M$, such that $K = C \oplus U$, then the metric projection $P_{M|K}$ admits a continuous, positively homogeneous and additive selection ([4], Theorem A).

This condition is not necessary for the existence of such a selection. In this paper we shall give a reformulation (called Theorem A') of Theorem A from [4] and prove that if $P_{M|K}$ admits a continuous, positively homogeneous and additive selection, satisfying some suplementary conditions, then the cone K admits a decompositions $K = C \oplus U$, with $C \subset \operatorname{Ker} P_{M|K}$ and $U \subset M$, closed convex cones (Theorem B). Although the conditions in theorems A' and B are very close to be necessary and sufficient for the existence of a continuous, positively homogeneous and additive selection for $P_{M|K}$, we weren't able to find such conditions. Some situations which may occur are illustrated by some examples following Theorem B.

By a *convex cone* in X we understand a nonvoid subset K of X such that:

a) $x_1 + x_2 \in K$, for all $x_1, x_2 \in K$, and

b) $\lambda \cdot x \in K$, for all $x \in K$, and $\lambda \geq 0$.

A carefull examination of the statement and of the proof of Theorem A in [4] yields the following more detailed reformulation:

THEOREM A'. *Let M be a closed linear subspace of a normed space X and K a closed convex cone in X. If there exist two closed convex cones $C \subset \operatorname{Ker} P_{M/K}$ and $U \subset M$ such that $K = C \oplus U$, then the application $p : K \to M$, defined by $p(x) = z$, for $x = y + z \in K$, $y \in C$, $z \in U$, is a continuous, positively homogeneous and additive selection of the metric projection $P_{M|K}$. The subspace M is K-proximinal and $C = p^{-1}(0)$, $U = p(K)$.*

The following theorem shows that, in some cases, the existence of a continuous, positively homogeneous and additive selection for $P_{M|K}$ implies the decomposability of K in the form $K = C \oplus U$, with C and U closed convex cones.

THEOREM B. *Let X be a normed space, M a closed subspace of X and K a closed convex cone in X. Suppose that the metric projection $P_{M|K}$ admits a continuous, positively homogeneous and additive selection p such that:*

a) $p(K)$ is closed and contained in K, and

b) $x - p(x) \in K$, for all $x \in K$.

Then $p^{-1}(0)$ and $p(K)$ are closed convex cones contained in $\operatorname{Ker} P_{M|K}$ *and M respectively, and $K = p^{-1}(0) \oplus p(K)$.*

If $p(K)$ is a closed subspace of K or $M \subset K$ then the conditions a) and b) are automatically fulfilled.

Proof. By the additivity, positive homogeneity of p and the fact that K is a convex cone, it follows immediately that $p(K)$ is a convex cone contained in M. By hypothesis *a)* it is also closed.

By the continuity of p the set $p^{-1}(0) \subset \operatorname{Ker} P_{M|K}$ is closed. If $y \in p^{-1}(0)$ and $\lambda \geq 0$ then $p(\lambda \cdot y) = \lambda \cdot p(y) = 0$, showing that $\lambda \cdot y \in p^{-1}(0)$. Similarly, $y_1, y_2 \in p^{-1}(0)$ and the additivity of p imply $p(y_1 + y_2) = p(y_1) + p(y_2) = 0$, showing that $p^{-1}(0)$ is a closed convex cone contained in $\operatorname{Ker} P_{M|K}$.

Now we prove that $K = p^{-1}(0) + p(K)$. If $x \in K$ then by Condition *b)*, $y := x - p(x) \in K$. By Condition a), $p(x) \in K$ implying $x = y + p(x)$ with $y, p(x) \in K$. Using the additivity of the function p and the fact that $p(p(x)) = p(x)$ (in fact $p(m) = m$ for all $m \in M$) we obtain $p(x) = p(y) + p(p(x)) = p(y) + p(x)$. It follows $p(y) = 0$, i.e. $y \in p^{-1}(0)$ and $K \subset p^{-1}(0) + p(K)$. Since $p^{-1}(0)$ and $p(K)$ are contained in K and K is a convex cone, it follows that $p^{-1}(0) + p(K) \subset K$ and $K = p^{-1}(0) + p(K)$.

To show that this is a direct algebraic sum, suppose that an element $x \in K$ admits two representations: $x = y + p(x)$ and $x = y' + z'$, with $y, y' \in p^{-1}(0)$ and $z' \in p(K) \subset M$. It follows $p(z') = z'$ and, by the additivity of p, $p(x) = p(y') + p(z') = 0 + z' = z'$, implying $y' = x - p(x) = y$ and $z' = p(x)$.

It remains to show that the correspondence $(y,z) \to y+z$, $y \in p^{-1}(0)$, $z \in p(K)$, is a homeomorphism between $p^{-1}(0) \times p(K)$, equipped with the product topology, and K. To this end consider a sequence $(y_n, z_n) \in p^{-1}(0) \times p(K)$, $n \in N$, converging to $(y, z) \in p^{-1}(0) \times p(K)$. It follows $y_n \to y$ and $z_n \to z$, implying $(y_n, z_n) \to y + z$, which proves the continuity of the application $(y, z) \to y + z$.

To prove the continuity of the inverse application $x \mapsto (y, z)$, where $x = y + z$, $y \in p^{-1}(0)$, $z \in p(K)$, take again a sequence $x_n = y_n + z_n \in K$, $y_n \in p^{-1}(0)$, $z_n \in p(K)$, converging to $x = y + z \in K$, where $y \in p^{-1}(0)$ and $z \in p(K)$. It follows $z_n = p(x_n)$, $n \in N$, $z = p(x)$, and, by the continuity of the application p, $z_n = p(x_n) \to p(x_n) = z$. But then $y_n = x_n - z_n \to x - z = y$, proving that the sequence $((y_n, z_n))_{n \in N}$ converges to (y, z) with respect to the product topology of $p^{-1}(0) \times p(K)$. This shows that the application $x \mapsto (y, z)$, $x = y + z \in p^{-1}(0) + p(K)$, is continuous too and, consequently, the application $(y, z) \mapsto y + z$ is a homeomorphism between $p^{-1}(0) \times p(K)$ and K.

If $p(K)$ is a closed subspace of K then Condition *a)* holds and, for $x \in K$, $p(x)$ and $-p(x)$ are in $p(K) \subset K$ so that $x - p(x) \in K$, showing that Condition *b)* holds too. If $M \subset K$ then $M = p(M) \subset p(K)$ and, since $p(K) \subset M$, it follows that $p(K) = M$ is a closed subspace of K. Theorem B is completely proved.

Remark. Conditions *a)* and *b)* are fulfilled by the selection p given in Theorem A'. Indeed, $K = p^{-1}(0) \oplus p(K)$ implies that $p(K)$ is a closed convex cone contained in K. Since every $x \in K$ can be written in the form $x = y + z$ with $p(y) = 0$ and $z = p(x)$, it follows that $x - p(x) = x - z = y \in K$ for all $x \in K$.

In the following examples, there always exists a continuous, positively homogeneous and additive selection of the metric projections but, the equality $K = p^{-1}(0) \oplus p(K)$ is not true in all these cases.

Example 1. Take $X = R^2$ with the Euclidean norm and $M\{(x_1, 0): x_1 \in R\}$. Then $P_M((x_1, x_2)) = \{(x_1, 0)\}$, for all $(x_1, x_2) \in R^2$, i.e. M is a Chebyshevian subspace of X and the only selection of the metric projection is $p((x_1, x_2)) = (x_1, 0)$, for $(x_1, x_2) \in R^2$. Let $R^2_+ := \{(x_1, x_2) \in R^2 : x_1 \geq 0, x_2 \geq 0\}$.

a) Take $K = \{(x_1, x_2): x_1 = x_2, x_1 \geq 0\}$. In this case $\operatorname{Ker} P_{M|K} = \{(0,0)\}$ so that the only closed convex cone contained in $\operatorname{Ker} P_{M|K}$ is $C = \{(0,0)\}$. The subspace M contains two nontrivial closed cones $U_+ = \{(x_1, 0): x_1 \geq 0\}$ and $U_- = \{(x_1, 0): x_1 \leq 0\}$ $p(K) = U_+$ and $K \neq C \oplus U_+ = U_+$.

b) Let $K = \{(x_1, x_2) \in R^2 : x_2 \geq x_1, x_1 \geq 0\}$. In this case $\operatorname{Ker} P_{M|K} = \{(0, x_2): x_2 \geq 0\}$ and the only nontrivial closed convex cone contained in $\operatorname{Ker} P_{M|K}$ is $C = \operatorname{Ker} P_{M|K}$. Again $p(K) = U_+$ but $K \neq C \oplus p(K) = R^2_+$.

c) Let $K = \{(x_1, x_2): x_2 \leq x_1, x_1 \geq 0\}$. In this case $\operatorname{Ker} P_{M|K} = \{(0,0)\}$ implying $C = \{(0,0)\}$. We have $p(K) = U_+ \subset K \cap M$ but $K \neq C \oplus p(K) = R^2_+$.

d) $K = \{(x_1, x_2): x_1 \geq 0,\ x_2 \geq 0\}$. In this case $\operatorname{Ker} P_{M|K} = \{(0, x_2): x_2 \geq 0\}$, $C = p^{-1}(0) = \operatorname{Ker} P_{M|K}$, $p(K) = U_+$ and $K = p^{-1}(0) \oplus p(K)$.

e) $K = \{(x_1, x_2) \in R^2 : x_2 \geq 0\}$. In this case $p(K) = M \subset K$, $\operatorname{Ker} P_{K|M} = \{(0, x_2): x_2 \geq 0\}$ and $K = C \oplus p(K)$, where $C = \operatorname{Ker} P_{M|K}$.

Remarks. In Example 1.a) none of the Condition *a)* and *b)* from Theorem A' is verified.

In Example 1.b) condition *b)* is fulfilled but $p(K) \not\subset K$, while in Example 1.c), $p(K) \subset K$ but $x - p(x) \in K$ only for $x = (0,0)$.

In Example 1.d) Conditions *a)* and *b)* are both verified but $p(K)$ is not a subspace of K.

In Example 1.e) $p(K) = M$.

The following example shows that $p(K)$ may be a closed subspace of K with $p(K) \neq M$.

Example 2. Let $X = R^3$ with the Euclidean norm, $M = \{(x_1, x_2, 0): x_1, x_2 \in R\}$ and $K = \{(0, x_2, x_3): x_2 \in R, x_3 \geq 0\}$. Then $p(K) = \{(0, x_2, 0): x_2 \in R\} \neq M$ and $\operatorname{Ker} P_{M|K} = \{(0, x_3, 0): x_3 \geq 0\}$. The equality $K = C \oplus p(K)$ holds with $C = \operatorname{Ker} P_{M|K}$.

Example 3. Let $X = C[a,b]$ be the Banach space of all continuous real-valued functions on the interval $[a,b]$ with the sup-norm.

The set

$$M := \{f \in C[a,b] : f(a) = f(b) = 0\}$$

is a closed subspace of $C[a,b]$,

$$K := \{f \in C[a,b] : f(a) = f(b) \geq 0\},$$

is a closed convex cone in $C[a,b]$ and $M \subset K$.

First show that the subspace M is K-proximinal. For $f \in K$, the function g defined by $g(x) := f(x) - f(a)$, $x \in [a,b]$, is an element of best approximation for f in K. Indeed, we have $\|f-g\| = f(a)$ and $\|f-h\| \geq |f(a)-h(a)|$, for all $h \in M$. It follows that $d(f,M) = f(a)$ and $g \in P_{M|K}(f)$.

The kernel of the restricted metric projection is

$$\operatorname{Ker} P_{M|K} = \{f \in K : 0 \in P_{M|K}(f)\} = \\ = \{f \in K : -f(a) \leq f(x) \leq f(a), \text{ for all } x \in [a,b]\}.$$

It follows $p(f) \in P_{M|K}(f)$ and the inequalities

$$\|p(f_1) - p(f_2)\| \leq \|f_1 - f_2\| + |f_1(a) - f_2(a)| \leq 2 \cdot \|f_1 - f_2\|,$$

for $f_1 f_2 \in K$, imply the continuity of the application p.

Obviously that p is positively homogeneous and additive on K. Since $M \subset K$, Theorem B can be applied to obtain the equality $K = p^{-1}(0) \oplus p(K)$. In this case $p^{-1}(0) = \{g \in K : \exists c \geq 0, g(x) = c, \text{ for all } x \in [a,b]\}$ and $f(x) = f(a) + (f(x) - f(a))$ is the unique decomposition of $f \in K$ in the form $f = g + h$ with $g \in p^{-1}(0)$ and $h \in p(K)$ $(g(x) = f(a)$ and $h(x) = f(x) - f(a)$ for all $x \in [a,b])$.

In Examples 1d) and e), the subspace M is K-Chebyshevian and $K = \operatorname{Ker} P_{M|K} \oplus p(K)$. The following corollary shows that this is a general property of K-Chebyshevian subspaces.

COROLLARY 1. *Let K be a closed convex cone in the normed space X and M a K-Chebyshevian subspace of X. If there exist two closed convex cones $C \subset \operatorname{Ker} P_{M|K}$ and $U \subset M$ such that $K = C \oplus U$, then $C = \operatorname{Ker} P_{M|K}$ and $U = p(K)$ where $p : K \to M$ is the only selection associated to the metric projection $P_{M|K}$.*

Proof. Since $C \subset \operatorname{Ker} P_{M|K}$ it remains to show that $\operatorname{Ker} P_{M|K} \subset C$. Let $x \in \operatorname{Ker} P_{M|K}$ and let $y \in C$, $z \in U$ be such that $x = y + z$. By Theorem A' the selection p is given by $p(x) = z$ and by the additivity of p.

$$0 = p(x) = p(y) + p(z) = 0 + z = z,$$

implying $x = y \in C$. The equality $U = p(K)$ follows also from Theorem A'.

A partial converse of Corollary 1 is also true:

COROLLARY 2. *Let M be a closed subspace of the normed space X and K a closed convex cone in X. If $K = \operatorname{Ker} P_{M|K} \oplus M$ then the subspace M is K-Chebyshevian and the only selection associated to the metric projection is continuous, positively homogeneous and additive on K.*

Proof. First we prove that $P_{M|K}(y) = \{0\}$ for every $y \in \operatorname{Ker} P_{M|K}$. Indeed, $y \in \operatorname{Ker} P_{M|K}$ is equivalent to $0 \in \operatorname{Ker} P_{M|K}(y)$. If $z \in \operatorname{Ker} P_{M|K}(y)$ then, taking into account the fact that M is a subspace of X and $z \in M$, we obtain

$$\|y - z\| = \inf\{\|y - m\| : m \in M\} = \\ = \inf\{\|y - z - m'\| : m' \in M\} = d(y - z, M),$$

showing that $0 \in P_{M|K}(y - z)$ or, equivalently, $y - z \in \operatorname{Ker} P_{M|K}$. But then y admits two representations $y = y + 0$ and $y = (y - z) + z$, with y, $y - z \in \operatorname{Ker} P_{M|K}$ and $0, z \in M$. The unicity of this representation implies $z = 0$.

Now, writing an arbitrary element $x \in K$ in the form $x = y + z$, with $y \in \operatorname{Ker} P_{M|K}$ and $z \in M$, we obtain

$$P_{M|K}(x) = P_{M|K}(y + z) = z + P_{M|K}(y) = z + 0 = z,$$

showing that z is the only element of best approximation for x in M, i.e. the subspace M is K-Chebyshevian.

We conclude the paper by an example of a non-Chebyshevian subspace of $\mathbb{R}^2$ for which the decomposition $K = C \oplus M$ is true.

Example 4. Let $X = R^2$ with the sup-norm

$$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}, \quad (x_1, x_2) \in R^2,$$

and

$$M := \{(x_1, 0) : x_1 \in R\},$$
$$K := \{(x_1, x_2) : x_1 \in R,\ x_2 \geq 0\}.$$

It is easily seen that

$$P_{M|K}((x_1, x_2)) = \{(m, 0) : x_1 - x_2 \leq m \leq x_1 + x_2\}, \text{ for } (x_1, x_2) \in K.$$

Indeed, $x_1 - x_2 \leq m \leq x_1 + x_2$ is equivalent to $|x_1 - m| \leq x_2$, implying

$$\|(x_1, x_2) - (m, 0)\| = \|x_1 - m, x_2\| = x_2.$$

If $(m', 0)$ is an arbitrary element of M then

$$\|(x_1, x_2) - (m', 0)\| = \max\{|x_1 - m'|, x_2\} \geq x_2,$$

showing that $d((x_1, x_2), M) = x_2$ and $(m, 0) \in P_{M|K}((x_1, x_2))$ if and only if $m \in R$ verifies the inequality $|x_1 - m| \leq x_2$.

The kernel of $P_{M|K}$ is

$$\text{Ker}\, P_{M|K} = \{(x_1, x_2) \in R^2 : |x_1| \leq x_2,\ x_2 \geq 0\},$$

and $K = C \oplus M$, where $C = \{(0, x_2) : x_2 \geq 0\}$ is a closed convex cone strictly contained in $\text{Ker}\, P_{M|K}$.

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Received 3 VIII 1994

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