REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 24, N^{os} 1–2, 1995, pp. 53–57

A THIRD ORDER AVERAGING THEOREM FOR KBM FIELDS

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The averaging theory is one of the most powerful tools in approaching problems governed by differential equations. The goal of this note is to present a theoretical extension of the averaging method (based on important works in this domain: [1-3]), materialized into a third order averaging theorem for differential systems having fields with the Krylov-Bogolyubov-Mitropolskij (KBM) property.

The theoretical results we shall present here were developed as a consequence of the practical necessities following from problems belonging mainly to celestial mechanics (and space dynamics), but not only. Our theorem and its corollary (for the case of periodic fields) describe constructive methods for obtaining approximate solutions for the considered differential systems; this recommends them for numerical applications. Their domain of applicability is very large, transcending considerably the celestial mechanics.

DEFINITION 1. Let z be a small positive real parameter, let $t \in [0,\infty)$ be a time-type variable, and let $x \in D \subset \mathbf{R}^n$ be an n-dimensional (spatial-type) vector. Let $a:[0,\infty) \times D \to \mathbf{R}^n$ be a KBM function of average a^0. Then we define the operator

$$\mathbf{A}(a)(z,t,x) := z\int_0^t \left[a,(s,x) - a^0(x)\right]\mathrm{d}s, \tag{1}$$

and denote

$$\|\mathbf{A}(a)\|_z := \sup_{0 \le zt < 1,\, x \in D} |\mathbf{A}(a)(z,t,x)| . \tag{2}$$

DEFINITION 2. Let a be the function considered in Definition 1, consider $b:[0,\infty) \times D \to \mathbf{R}^n$, and suppose that a and b admit spatial derivatives (*i.e.* with respect to the components of x). Then we define the operator

$$\mathbf{B}(a,b)(t,x) := \nabla a(t,x) \cdot b(t,x) - \nabla b(t,x) \cdot a^0(x). \tag{3}$$

Let now $f, g, h: [0, \infty) \times D \to \mathbf{R}^n$ be continuous functions with respect to all variables and having the properties: f admits a uniformly Lipschitzian spatial second order derivative, g admits a uniformly Lipschitzian spatial first order derivative, and h is uniformly Lipschitzian with respect to its spatial variables. Consider the initial value problem

$$\text{(4)} \qquad \begin{aligned} &\mathrm{d}x / \mathrm{d}t = zf(t,x) + z^2 g(t,x) + z^3 h(t,x), \\ &x(0) = x_0. \end{aligned}$$

Let $u(t)$ be the solution of the initial value problem

$$\text{(5)} \qquad \begin{aligned} \mathrm{d}u / \mathrm{d}t = {} & zf^0(u) + z^2 g^0(u) + z^3 h^0(u) + zd_1(z) f^{10}(u) + \\ & + zd_1(z) d_2(z) f^{20}(u) + z^2 d_3(z) f^{30}(u) + \\ & + z^2 d_1(z) g^{10}(u) + zd_1^2(z) f^{40}(u), \\ u(0) = {} & x_0, \end{aligned}$$

where we have written

$$\text{(6)} \qquad \begin{aligned} & f^i(t,x) := \mathbf{B}(f, u^i)(t,x), \quad i = \overline{1,3}, \\ & g^1(t,x) := \mathbf{B}(g, u^1)(t,x), \\ & f^4(t,x) := \frac{1}{2} u^1(t,x) \cdot \nabla\nabla f(t,x) \cdot u^1(t,x) - \nabla u^1(t,x) \cdot f^{10}(x), \end{aligned}$$

and considered that the functions $f, g, h,$ and those defined by (6) are KBM functions of averages $f^0, g^0, h^0, f^{10}, f^{20}, f^{30}, g^{10}, f^{40}$, respectively. We also wrote

$$\text{(7)} \qquad \begin{aligned} & d_1(z) u^1(t,x) := \mathbf{A}(f)(z,t,x), && d_1(z) := \|\mathbf{A}(f)\|_z, \\ & d_2(z) u^2(t,x) := \mathbf{A}(f^1)(z,t,x), && d_2(z) := \|\mathbf{A}(f^1)\|_z, \\ & d_3(z) u^3(t,x) := \mathbf{A}(g)(z,t,x), && d_3(z) := \|\mathbf{A}(g)\|_z, \end{aligned}$$

and

$$\text{(8)} \qquad d_4(z) := \left\| \mathbf{A}\left(g^1 + h + f^2 + f^3 + f^4\right) \right\|_z .$$

Under these conditions the following theorem holds:

THEOREM 1 *(general third order averaging theorem): If the solution $u(t)$ of problem (5) belongs to the interior of the domain D on a time scale of order z^{-1}, then the solution of problem (4) can be written as*

$$\text{(9)} \qquad \begin{aligned} x(t) = {} & u(t) + d_1(z) u^1(t, u(t)) + d_1(z) d_2(z) u^2(t, u(t)) + \\ & + zd_3(z) u^3(t, u(t)) + \mathrm{O}\big[(d_1(z)(d_1(z) + d_2(z) + z) + \\ & + zd_3(z))\left(\sqrt{d_4(z)} + d_1(z) + z\right) + z^2 \sqrt{d_4(z)}\big] \end{aligned}$$

on a time scale of order z^{-1}.

Proof. Consider the following change of variable

$$\text{(10)} \qquad x = y + d_1(z) u^1(t,y) + d_1(z) d_2(z) u^2(t,y) + zd_3(z) u^3(t,y).$$

Differentiating (10) with respect to t, and taking into account (4), (6), and (7), a straightforward calculation yields the following approximation of the system whose solution is y:

$$\text{(11)} \qquad \begin{aligned} \mathrm{d}y / \mathrm{d}t = {} & zf^0(y) + z^2 g^0(y) + zd_1(z) f^{10}(y) + z^3 h(t,y) + \\ & + z^2 d_1(z) g^1(t,y) + zd_1(z) d_2(z) f^2(t,y) + \\ & + z^2 d_3(z) f^3(t,y) + zd_1^2(z) f^4(t,y) + \\ & + O\big[z(z + d_1(z))(d_1(z)(d_1(z) + d_2(z) + z) + zd_3(z))\big], \\ y(0) = {} & x_0. \end{aligned}$$

Consider the associated problem

$$\text{(12)} \qquad \begin{aligned} \mathrm{d}v / \mathrm{d}t = {} & zf^0(v) + z^2 g^0(v) + zd_1(z) f^{10}(v) + z^3 h_T(t,v) + \\ & + z^2 d_1(z) g_T^1(t,v) + zd_1(z) d_2(z) f_T^2(t,v) + \\ & + z^2 d_3(z) f_T^3(t,v) + zd_1^2(z) f_T^4(t,v), \\ v(0) = {} & x_0. \end{aligned}$$

where subscript T marks the following type of averaging:

$$\text{(13)} \qquad q_T(t,v) := T^{-1} \int_t^{t+T} q(s,v) \mathrm{d}s.$$

Using the well-known estimate [3]

$$\text{(14)} \qquad \int_0^t q_T(s, v(s)) \mathrm{d}s = \int_0^t q(s, z(s)) \mathrm{d}s + O(T),$$

and Gronwall's lemma, and observing the hypothesis according to which the fields is Lipschitzian, we get an approximation of the difference between the solutions of systems (11) and (12):

$$\text{(15)} \qquad \begin{aligned} y(t) - v(t) = O\big[& (z + d_1(z))\big(d_1(z)(d_1(z) + d_2(z) + z) + zd_3(z)\big) + \\ & + z\left(d_1(z)(d_1(z) + d_2(z) + z) + zd_3(z) + z^2\right) T \big]. \end{aligned}$$

By virtue of the known estimate [3]

$$\text{(16)} \qquad q_T(s, v(s)) = q^0(v(s)) + O\big[\|\mathbf{A}(q)\|_z / (zT)\big],$$

and applying again Gronwall's lemma (under the same hypothesis about the field), the difference between the solutions of problems (12) and (5) can be approximated as

$$(17)\quad v(t)-u(t)=O\left[d_4(z)\left(d_1(z)\left(d_1(z)+d_2(z)+z\right)+zd_3(z)+z^2\right)/(zT)\right].$$

Lastly, choosing T such that $(zT)^2=d_4(z)$, and applying the triangle inequality, (15) and (17) lead to

$$(18)\quad y(t)-u(t)=O\Big[\left(d_1(z)\left(d_1(z)+d_2(z)+z\right)+zd_3(z)\right)\left(\sqrt{d_4(z)}+\right.$$
$$\left.+d_1(z)+z\right)+z^2\sqrt{d_4(z)}\Big].$$

If $u(t)$ belongs to the interior of the domain D on a time scale of order z^{-1}, then from (10) and (18) one obtains (9) and the theorem is proved.

Now, consider again problem (4), all conditions of Theorem 1 being fulfilled, and in addition f, g, h, are periodic in t. Let $u(t)$ be the solution of the problem

$$(19)\quad \begin{aligned} du/dt &= z\bar{f}^0(u)+z^2\bar{f}^{10}(u)+z^3\bar{f}^{20}(u),\\ u(0)&=x_0,\end{aligned}$$

where

$$(20)\quad \begin{aligned} f^1(t,x)&:=g(t,x)+\mathbf{B}(f,u^1)(t,x),\\ f^2(t,x)&:=h(t,x)+\mathbf{B}(f,u^2)(t,x)+\mathbf{B}(g,u^1)(t,x)+\\ &+\frac{1}{2}u^1(t,x)\cdot\nabla\nabla f(t,x)\cdot u^1(t,x)-\nabla u^1(t,x)\cdot\bar{f}^{10}(x),\end{aligned}$$

and

$$(21)\quad \begin{aligned} zu^1(t,x)&:=\mathbf{A}(f)(z,t,x),\\ zu^2(t,x)&:=\mathbf{A}(f^1+g)(z,t,x),\end{aligned}$$

while $\bar{f}^0,\bar{f}^{10},\bar{f}^{20}$ stand for the averages of the respective fields over one period.

Under these conditions, the following result holds:

COROLLARY 1 (periodic case). *If the solution $u(t)$ of problem (19) belongs to the interior of the domain D on a time scale of order z^{-1}, then the solution of problem (4), in which f, g, h are periodic in t, can be written as*

$$(22)\quad x(t)=u(t)+zu^1(t,u(t))+z^2u^2(t,u(t))+O(z^3)$$

on a time scale of order z^{-1}.

The proof is entirely analogue to that of Theorem 1, in which the supplementary hypothesis of periodicity of the field is used.

Acknowledgements. The authors wish to express their gratitude to Professor E. A. Grebenikov for reading the manuscript and for valuable comments and advice. Fruitful discussions with their colleague D. Şelaru and with Dr. V. Mioc are also acknowledged.

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Received 1 VI 1994

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