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# THE NUMERICAL TREATMENT OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND BY THE EXACT COLLOCATION METHOD

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## 1. INTRODUCTION

Consider the nonlinear Volterra integral equation of the second kind:

$$(1.1)\qquad y(t) = f(t) + \int_0^t K(t, s, y(s))\,\mathrm{d}s,\quad t \in I := [0, T],$$

where the given functions $f$ and $K$, defined on $I$, resp. $S \times \mathbb{R}$, ($S := \{(t, s): 0 \le s \le t \le T\}$), are supposed to be sufficiently smooth for the integral equation (1. 1) to have a unique solution $y \in C^{\alpha}(I)$, with $\alpha \in \mathbb{N}$ (see [2], [5]).

The problem of determining an approximate solution for the integral equation (1.1) using the collocation method has been studied a lot (see [2] and the bibliography cited therein).

The method described in the following uses as approximating space the space of polynomial spline functions.

Let $\Pi_N: 0 = t_0 < \ldots < t_N = T$ ( with $t_n = t_n^{(N)}$, $N \le 1$) be a mesh for $I$, and we shall use the following notation (see [1]):

$$h_n := t_{n+1} - t_n,\ n = 0, 1, \ldots, N-1;$$
$$h := \max\{h_n : n = 0, 1, \ldots, N-1\};$$
$$\sigma_0 := [t_0, t_1],\ \sigma_n := (t_n, t_{n+1}],\ n = 1, 2, \ldots, N-1;$$
$$Z_N := \{t_n : n = 1, 2, \ldots, N-1\};$$

and $Z_N := Z_N \cup \{T\}$.

For a fixed $N \ge 1$ and for all $m, d \in \mathbf{Z}$ with $m \ge 1$ and $d \ge -1$ the space of polynomial spline functions of degree $m+d$ and continuity class $d$, possessing the knots $Z_N$ is, by definition, the set:

$$S_{m+d}^{(d)} := \left\{ u : u(t)\big|_{t\in\sigma_n} =: u_n(t) \in \mathcal{P}_{m+d}, n = 0,\ldots,N-1; \right.$$

$$\left. u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } j = 0,1,\ldots,d \text{ and } t_n \in Z_n \right\},$$

whose elements, on each subinterval $\sigma_n$, are polynomials of degree not greater than $m+d$, and which in the knots $Z_N$ are of continuity class $d$. If $d=-1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinuities at the knots $Z_N$.

In the extreme cases, when $m \geq 1$ and $d=-1$, i.e in $S_{m-1}^{(-1)}(Z_N)$, the problem of approximating the exact solution of (1.1) has been solved by H. Brunner and P.J. van der Houwen (see [2]), and when $d \in \mathbb{N}$ and $m=1$, i.e. in $S_{d+1}^{(d)}(Z_N)$, an approximate solution has been constructed by M. Micula and G. Micula in [6], whose convergence to the exact solution has been proved by Malina Lubor in [4].

In this paper we shall construct an approximate solution for the exact solution of integral equation (1.1) in the space of polynomial spline functions $S_{m+d}^{(d)}(Z_N)$, with $m \geq 1$ and $d \geq -1$ and we shall prove its convergence to the exact solution. Moreover, we shall study the conditions under which we can obtain local superconvergence results. In the end some numerical examples are presented.

## 2. THE EXACT COLLOCATION EQUATION IN $S_{m+d}^{(d)}(Z_N)$

We shall assume in the following that the mesh sequence $(\Pi_N)_{N\geq 1}$ is quasiuniform, that is, there exists a finite constant $\gamma$ independent of $N$ such that

$$\max_{(n)} h_n / \min_{(n)} h_n \leq \gamma < \infty, \text{ for all } N \in \mathbb{N}.$$

In [6], M. Micula and G. Micula proved that an element $u \in S_{m+d}^{(d)}(Z_N)$ has for all $n = 0, 1,\ldots, N-1$ and for all $t \in \sigma_n$ the following form:

$$u(t) = u_n(t) = \sum_{r=0}^{d} \frac{u_{n-1}^{(r)}(t_n)}{r!}(t-t_n)^r + \sum_{r=1}^{m} a_{n,r}(t-t_n)^{d+r}; \tag{2.1}$$

where

$$u_{n-1}^{(r)}(t_n) := \left[\frac{\mathrm{d}^r}{\mathrm{d}t^r} u_{n-1}(t)\right]_{t=t_n}, \quad r = 0,1,\ldots,d;$$

and

$$u_{-1}^{(r)}(0) := \left[\frac{\mathrm{d}^r}{\mathrm{d}t^r} y(t)\right]_{t=0} = y_{(0)}^{(r)}, \quad r = 0,1,\ldots,d.$$

From (2.1) we have that an element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined when we know the coefficients $\{a_{n,r}\}_{r=\overline{1,m}}$ for all $n = 0,1,\ldots,N-1$. In order to compute



these coefficients we consider the set of collocation parameters $\{c_j\}_{j=\overline{1,m}}$, where $0 < c_1 < \ldots < c_m \leq 1$ and the set of collection points is defined by:

$$X(N) := \bigcup_{n=0}^{N-1} X_n, \tag{2.2a}$$

where

$$X_n := \{t_{n,j} := t_n + c_j h_n : j = 1,2,\ldots,m\}, \quad n = 0,1,\ldots,N-1. \tag{2.2b}$$

So, the approximate solution $u \in S_{m+d}^{(d)}(Z_N)$ will be determined imposing the condition that $u$ satisfy the integral equation (1.1) on the finite set $X(N)$,

$$u(t) = f(t) + \int_0^t K(t,s,u(s))\mathrm{d}s, \quad \text{for all } t \in X(N). \tag{2.3}$$

Equation (2.3) is called the exact collocation equation for the integral equation (1.1).

Using (2.2), equation (2.3) can be written:

$$u(t_{n,j}) = f(t_{n,j}) + h_n \int_0^{c_j} K(t_{n,j}, t_n + \tau h_n, u_n(t_n + \tau h_n))\mathrm{d}\tau + F(t_{n,j}) \tag{2.4a}$$

$$\text{for } j = 1,2,\ldots,m \text{ and } n = 0,1,\ldots,N-1,$$

where

$$F_n(t) := \sum_{i=0}^{n-1} h_i \int_0^1 K(t, t_i + \tau h_i, u_i(t_i + \tau h_i))\mathrm{d}\tau, \text{ for all } t \in X_n, \tag{2.4b}$$

denotes the lag term:

If we replace $u$ in (2.4) using (2.1) we obtain for all $n = 0,1,\ldots, N-1$

$$\sum_{r=0}^{d} \frac{u_{n-1}^{(r)}(t_n)}{r!} c_j^r h_n^r + c_j^d h_n^d \sum_{r=1}^{m} a_{n,r} c_j^r h_j^r = \tag{2.5a}$$

$$= f(t_{n,j}) + h_n \int_0^{c_j} k\left(t_{n,j}, t_n + \tau h_n, \sum_{r=0}^{d} \frac{u_{n-1}^{(r)}(t_n)}{r!}\tau^r h_n^r + \right.$$

$$\left. + \sum_{r=1}^{m} a_{n,r} \tau^{d+r} h_n^{d+r}\right)\mathrm{d}\tau + F_n(t_{n,j}), \quad \text{for } j = 1,2,\ldots,m,$$

where

$$F_n(t) := \sum_{i=0}^{n-1} h_i \int_0^1 K\left(t, t_i + \tau h_i, \sum_{r=0}^{d} \frac{u_{i-1}^{(r)}(t_i)}{r!}\tau^r h_i^r + \sum_{r=1}^{m} a_{i,r}\tau^{d+r} h_i^{d+r}\right)\mathrm{d}\tau \tag{2.5b}$$

One can observe that equation (2.5) represents, for each $n = 0,1,\ldots, N-1$, a recursive system which will give the unknowns $\{a_{n,r}\}_{r=\overline{1,m}}$. Since this solution has been found, the values of $u$ and its derivatives $u', \ldots, u^{(d)}$ on $\sigma_n$ are determined by formula (2.1).

*Remark* 2.1. One can prove, using the Banach fixed point principle, that for $h$ small enough system (2.5) has a unique solution and so the polynomial spline function $u \in S^{(d)}_{m+d}(Z_N)$, obtained from the above algorithm is uniquely determined (see [2], [5]).

Suppose now that the given functions $f$ and $K$ are of class $m+d+1$ on their domain of definition. This assumption implies the fact that the integral equation (1.1) has a unique solution $y$, which is also of class $m+d+1$. We denote $e^{(k)} = y^{(k)} - u^{(k)}$, $k = 0, 1, \ldots, m+d$ the approximation error of the exact solution $y$ and of its derivatives up to $k^{th}$ order by the approximate solution $u$ and, respectively, by its derivatives. We will denote by $e_n^{(k)}$ the restriction of $e^{(k)}$ to the subinterval $\sigma_n$ for all $n = 0,1,\ldots,N-1$ and $k = 0,1,\ldots, m+d$, and we shall use the following notation:

$$\left\|e^{(k)}\right\|_\infty := \sup\left\{\left|e_n^k(t)\right| : t \in \sigma_n,\ n = 0,1,\ldots,N-1\right\} \tag{2.6}$$

for all $k = 0,1,\ldots,m+d$.

Concerning the convergence of the method described above we give the following theorem:

THEOREM 2.2 *If* $f \in C^{m+d+1}(I)$ *and* $K \in C^{m+d+1}(S \times \mathbb{R})$ *in the nonlinear Volterra integral equation of the second kind* (1.1), *then*:

(i) *there exists* $\bar{h} > 0$ *such that for each* $h \in (0, \bar{h})$ *system* (2.5) *has a unique solution that determines a unique element* $u \in S^{(d)}_{m+d}(Z_N)$;

(ii) *for every choise of the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *with* $0 < c_1 < \ldots < c_m \le 1$ *and for all quasi-uniform mesh sequences we have*:

$$\left\|e^{(k)}\right\|_\infty \le C_k h^{m+d+1-k},\quad k = 0,1,\ldots,m+d, \tag{2.7}$$

*where* $C_k$, $k = 0,1,\ldots,m+d$ *denote some finite constants independent of* $h$ *(but depending on the* $\{c_j\}$*)*.

*Proof.* (i) It follows from Remark 2.1.

(ii) We shall prove it by induction.

First, we develop the exact solution $y$ in $\sigma_0 = [0, t_0]$ in Taylor series in the neighborhood of the origin, and we obtain for all $\tau \in [0,1]$, that:



$$y(\tau h_0) = \sum_{r=0}^{m+d} \frac{y^{(r)}(0)}{r!}\tau^r h_0^r + R_0(\tau)\cdot h_0^{m+d+1}, \tag{2.8}$$

where

$$R_0(\tau) := \frac{y^{(m+d+1)}(\xi_0)}{(m+d+1)!}\tau^{m+d+1},\quad 0 < \xi_0 < \tau h_0.$$

So, by (2.1) with $n = 0$, we have for all $\tau \in [0,1]$

$$e(\tau h_0) = y(\tau h_0) - u_0(\tau h_0) = h_0^{m+d+1}\left[\sum_{r=1}^{m} \beta_{0,r}\tau^{d+r} + R_0(\tau)\right], \tag{2.9a}$$

where

$$h_0^{m+1}\beta_{0,r} := \left(\frac{y^{(d+r)}(0)}{(d+r)!} - a_{0,r}\right)h_0^r,\ r = 1,2,\ldots,m. \tag{2.9b}$$

Now we shall prove that the coefficients $\{\beta_{0,r}\}_{r=\overline{1,m}}$ are bounded. Since $y$ is the exact solution of integral equation (1.1) and $u$ verifies collocation equations (2.3) for $\tau \in \{c_1, c_2, \ldots, c_m\}$, then for all $j = 1,2,\ldots,$ the following relations hold:

$$\begin{aligned} e(c_j h_0) &= h_0\int_0^{c_j}\left[K(c_j h_0, \tau h_0, y(\tau h_0)) - K(c_j h_0, \tau h_0, u_0(\tau h_0))\right]d\tau = \\ &= h_0\int_0^{c_j}\frac{\partial K}{\partial y}(c_j h_0, \tau h_0, z(\tau h_0))\cdot e(\tau h_0)\,d\tau, \end{aligned} \tag{2.10}$$

where $z(\tau h_0) \in [y(\tau h_0), u_0(\tau h_0)]$ for all $\tau \in [0, c_j]$.

From (2.9a) and (2.10) we obtain that

$$\begin{aligned} &\sum_{r=1}^{m}\beta_{0,r}\left[c_j^{d+r} - h_0\int_0^{c_j}\frac{\partial K}{\partial y}(c_j h_0, \tau h_0, z(\tau h_0))\tau^{d+r}d\tau\right] = \\ &= h_0\int_0^{c_j}\frac{\partial K}{\partial y}(c_j h_0, \tau h_0, z(\tau h_0))R_0(\tau)d\tau - R_0(c_j). \end{aligned} \tag{2.11}$$

holds, for all $j = 1,2,\ldots,m$.

Using the following notation:

$$\beta_0 := (\beta_{0,1}, \beta_{0,2}, \ldots, \beta_{0,n})^T,$$

$$R_0 := (R_0(c_1), R_0(c_2), \ldots, R_0(c_m))^T,$$

$$V := \left(c_j^{d+r}\right)_{j,r=\overline{1,m}},$$

$$D_0 := \left(\int_0^{c_j}\frac{\partial K}{\partial y}(c_j h_0, \tau h_0, z(\tau h_0))\tau^{d+r}d\tau\right)_{j,r=\overline{1,m}},$$

$$q_0 := (q_{0,1}, q_{0,2}, \ldots, q_{0,m})^T,$$

where

$$q_{0,j} := h_0 \int_0^{c_j} \frac{\partial K}{\partial y}\big(c_j h_0, \tau h_0, z(\tau h_0)\big) R_0(\tau) \mathrm{d}\tau.$$

System (2.11) can be written

(2.12) $$(V - h_0 D_0)\beta_0 = q_0 - R_0.$$

Since, $\det V \neq 0$ and $\left|\frac{\partial K}{\partial y}(t,s,y)\right| \leq K_0$ for all $(t,s,y) \in S \times \{y \in \mathbb{R} : |y| \leq b\}$, $K_0$ being a positive constant, it follows that there exists $\bar{h}_0 > 0$ such that the matrix $V - h_0 D_0$ possesses a uniformly bounded inverse for all $h_0 \in (0, \bar{h}_0)$. So, by (2.12) it follows that

(2.13) $$\beta_0 = (V - h_0 D_0)^{-1}(q_0 - R_0).$$

Since $R_0(\tau)$ is bounded for all $\tau \in [0,1]$ we have that there exists $M_0 > 0$ such that $|R_0(\tau)| \leq M_0$ for all $\tau \in [0,1]$ and

$$\|R_0(\tau)\|_1 := \sum_{l=1}^{m} |R_0(c_l)| \leq m M_0 =: M_0'$$

and

$$\|q_0\|_1 := \sum_{l=1}^{m} |q_{0,l}| \leq m h_0 K_0 M_0 =: Q_0.$$

Using these estimations in (2.13) we obtain the following evaluation

(2.14) $$\|\beta_0\|_1 \leq \left\|(V - h_0 D_0)^{-1}\right\| (M_0' + Q_0) =: B,$$

which together with (2.9a) proves that:

(2.15) $$\sup\{|e(t)| : t \in \sigma_0\} \leq C_0 h_0^{m+d+1} \leq C_0 h^{m+d+1}.$$

To prove that $|e^{(k)}(t)| \leq C_{0,k} h^{m+d+1-k}$ for all $k = 1, 2, \ldots, m+d$ and all $t \in \sigma_0$ instead of (2.8) we shall use the Taylor series with the rest in the integral form:

$$y(\tau h_0) = \sum_{r=0}^{m+d} \frac{y^{(r)}(0)}{r!} \tau^r h_0^r + \int_0^{\tau h_0} \frac{y^{(m+d+1)}(\xi)}{(m+d)!} (\tau h_0 - \xi)^{m+d} \mathrm{d}\xi.$$

Now, from (2.1) and (2.9a) we obtain



$$\left|e^{(k)}(\tau h_0)\right| = \left|y^{(k)}(\tau h_0) - u^{(k)}(\tau h_0)\right| \leq$$

$$\leq h_0^{m+d+1-k} \cdot \left[\sum_{r=1}^{m} \frac{(r+d)!}{(r+d-k)!} |\beta_{0,r}| + \frac{y^{(m+d+1)}(\xi_0)}{(m+d+1-k)!}\right]$$

for each $k = 1, 2, \ldots, d$, where $\xi_0 \in (0, h_0)$ and

$$\left|e^{(k)}(\tau h_0)\right| \leq h_0^{m+d+1-k} \left[\sum_{r=k-d}^{m} \frac{(r+d)!}{(r+d-k)!} |\beta_{0,r}| + \frac{y^{(m+d+1)}(\xi_0)}{(m+d+1-k)!}\right]$$

for each $k = d+1, d+2, \ldots, m+d$ and for all $\tau \in [0,1]$.

Since $|\beta_{0,r}|$ is bounded for all $r = 1, 2, \ldots, m$ and $\left|y^{(m+d+1)}(\xi_0)\right|$ is bounded too, we have that:

(2.16) $$\left|e^{(k)}(t)\right| \leq C_{0,k} h_0^{m+d+1-k} \leq C_0 h^{m+d+1-k}, \text{ for all } t \in \sigma_0.$$

So, (2.15) and (2.16) prove that the theorem is valid for the subinterval $\sigma_0$.

Suppose now that the following estimations hold for all $j \leq n-1$

(2.17) $$|e_j(t)| = |e(t_j + \tau t_j)| \leq C_j h^{m+d+1}, \ t \in \sigma_j, \ \tau \in (0,1],$$

and

(2.18) $$\left|e_j^{(k)}(t)\right| = C_{j,k} h^{m+d+1-k}, \ t \in \sigma_j, k = 1, 2, \ldots, m+d.$$

We shall prove that (2.17) and (2.18) hold for $j = n$. Therefore we develop the exact solution $y$ in the interval $\sigma_n$ in Taylor series

(2.19) $$y(t_n + \tau h_n) = \sum_{r=1}^{m+d} \frac{y^{(r)}(t_n)}{r!} \tau^r h_n^r + h_n^{m+d+1} R_n(\tau),$$

where

$$R_n(\tau) := \frac{1}{(m+d)!} \int_0^{\tau} y^{(m+d+1)}(t_n + s h_n)(\tau - s)^{m+d} \mathrm{d}s,$$

for all $\tau \in [0,1]$.

In order to obtain the error estimation on the subinterval $\sigma_n$, we use (2.19) and (2.1). It follows that:

(2.20a) $$e_n(t_n + \tau h_n) = y(t_n + \tau h_n) - u_n(t_n + \tau h_n) =$$

$$= \sum_{r=0}^{d} \frac{y^{(r)}(t_n) - u_{n-1}^{(r)}(t_n)}{r!} \tau^r h_n^r + h_n^{m+d+1} \left[\sum_{r=1}^{m} \beta_{n,r} \tau^{r+d} + R_n(\tau)\right],$$

where

$$h_n^{m+1}\beta_{n,r} = \left(\frac{y^{(d+r)}}{(d+r)!} - a_{n,r}\right)h_n^r. \tag{2.20b}$$

Taking account that $y$ satisfies integral equation (1.1) and $u$ satisfies the collocation equation (2.3) for all $\tau \in \{c_1, c_2, \ldots, c_n\}$,, we have that for all $j = 1, \ldots, m$

$$\begin{aligned} e_n(t_n + c_j h_n) = h_n \int_0^{c_j} \frac{\partial K}{\partial y}(t_{n,j}, t_n + \tau h_n, z_n(t_n + \tau h_n))e_n(t_n + \tau h_n)\mathrm{d}\tau + \\ + \sum_{i=0}^{n-1} hi \int_0^1 \frac{\partial K}{\partial y}(t_{n,j}, t_i + \tau h_i, z_i(t_i + \tau h_i))e_i(t_i + \tau h_i)\mathrm{d}\tau, \end{aligned} \tag{2.21}$$

where the functions $z_i$, $i = 0, 1, \ldots, n$ have the property that $z_i(t) \in [y(t), u_i(t)]$, for all $t \in \sigma_i$.

From (2.20a) and (2.21) we obtain the following system:

$$\begin{aligned} (V - h_n \cdot D_{n,n})\beta_n = \sum_{i=0}^{n-1} h_i \left[\left(\frac{h_i}{h_n}\right)^{m+d+1} D_{n,i}\beta_i + \frac{1}{h_n^{m+d+1}} F_{n,i} \cdot E_i\right] + \\ + \frac{1}{h_n^{m+d+1}}(h_n F_{n,n} - W)E_n + q_n, \end{aligned} \tag{2.22}$$

where:

$V := (c_j^{r+d})_{j,r=\overline{1,m}}$;

$W := (c_j^r)_{j,r=\overline{1,m}}$;

$E_i := \left(h_i^r \dfrac{y^{(r)}(t_i) - u_{i-1}^{(r)}(t_i)}{r!}\right)_{r=\overline{0,d}}$, $i = 0, 1, \ldots n$;

$$D_{n,i} := \begin{cases} \left(\int_0^1 \frac{\partial K}{\partial y}(t_{n,j}, t_i + \tau h_i, z_i(t_i + \tau h_i))\tau^{d+r}\mathrm{d}\tau\right)_{j,r=\overline{1,m}}, & \text{if } 0 \le i \le n-1 \\ \left(\int_0^{c_j} \frac{\partial K}{\partial y}(t_{n,j}, t_n + \tau h_n, z_n(t_n + \tau h_n))\tau^{d+r}\mathrm{d}\tau\right)_{j,r=\overline{1,m}}, & \text{if } i = n; \end{cases}$$

$$F_{n,i} := \begin{cases} \left(\int_0^1 \frac{\partial K}{\partial y}(t_{n,j}, t_i + \tau h_i, z_i(t_i + \tau h_i))\tau^r\mathrm{d}\tau\right)_{j=\overline{1,m},\, r=\overline{0,d}}, & \text{if } 0 \le i \le n-1 \\ \left(\int_0^{c_j} \frac{\partial K}{\partial y}(t_{n,j}, t_n + \tau h_n, z_n(t_n + \tau h_n))\tau^r\mathrm{d}\tau\right)_{j=\overline{1,m},\, r=\overline{0,d}}, & \text{if } i = n; \end{cases}$$

$\beta_i := (\beta_{i,r})_{r=\overline{1,m}}$, $i = 0, 1, \ldots, n$;

$q_n := (q_{n,j})_{j=\overline{1,m}}$;



with

$$q_{n,j} := -R_n(c_j) + h_n \int_0^{c_j} \frac{\partial K}{\partial y}(t_{n,j}, t_n + \tau h_n, z_n(t_n + \tau h_n))R_n(\tau)\mathrm{d}\tau + \\ + \sum_{i=0}^{n-1} h_i \left(\frac{h_i}{h_n}\right) \int_0^1 \frac{\partial K}{\partial y}(t_{n,j}, t_i + \tau h_i, z_i(t_i + \tau h_i))R_i(\tau)\mathrm{d}\tau.$$

Since, $\det V \neq 0$ and $\left|\dfrac{\partial K}{\partial y}(t, s, y)\right| \le K_0$ for all $(t, s, y) \in S \times \{y \in \mathbb{R} : |y| \le b\}$ it follows that there exists $\bar{h}_n > 0$ such that for all $h_n \in (0, \bar{h}_n)$ the matrix $V - h_n D_{n,n}$ possesses a uniformly bounded inverse $(V - h_n D_{n,n})^{-1}$. We also note that the matrices $F_{n,i}$, $i = \overline{0,n}$ are bounded, so there exists a positive constant $F$ such that

$$\| h_n F_{n,n} - W \| \le F. \tag{2.23}$$

Now, from (2.17) and (2.18) we get

$$\begin{aligned} \left\| \frac{1}{h_n^{m+d+1}} E_i \right\|_1 := \sum_{r=0}^{d} \frac{h_i^{(r)}}{h_n^{m+d+1}} \frac{y^{(r)}(t_i) - u_{i-1}^{(r)}(t_i)}{r!} \le \\ \le \sum_{r=0}^{d} \frac{C_{i,r}}{r!} \gamma^{m+d+1} := E^{(i)}. \end{aligned} \tag{2.24}$$

For the estimation of vector $q_n$ we use the fact $|R_n(\tau)| \le M_n$ ($M_n$ being a positive constant) for all $\tau \in (0, 1]$ and we obtain

$$\|q_n\|_1 := \sum_{j=1}^{m} |q_{n,j}| \le m(M_n + h_n K_0 M_n + (N-1)\gamma^{m+d+1} K_0 M_n h_i) := Q.$$

From (2.22), (2.23), (2.24) and from the above considerations it follows that:

$$\begin{aligned} \|\beta_n\|_1 \le h\gamma^{m+d+1} D_1 \left\|(V - h_n D_{n,n})^{-1}\right\| \sum_{i=1}^{n-1} \|\beta i\|_1 + \left\|(V - h_n D_{n,n})^{-1}\right\| \times \\ \times \left\{ h \sum_{i=0}^{n-1} \|F_{n,i}\| E_i + F E^{(n)} + Q \right\}, \end{aligned} \tag{2.25}$$

which represents a discrete Gronwall inequality for the $l_1$-norms of the vector $\beta_i$, $i = 0, 1, \ldots, n$. In this relation we have denoted $D_1 := \| D_{n,i} \|_1$, which is a finite norm. Inequality (2.25) proves that $\|\beta_n\|_1$ is bounded, i.e. there exists a positive number $B_n$ such that $\|\beta_n\|_1 < B_n$. Now, from (2.20a) it results

(2.26) $$|e_n(t_i+\tau h_i)| \le \left(E^{(n)}+B_n+M_n\right)h^{m+d+1} := C_n h^{m+d+1}$$

for all $\tau \in (0,1]$.

To prove (2.18) for $j=n$, we derive relations (2.1) and (2.19) $k$ times, $k=1,\dots,m+d$ and we obtain the following error estimations:

$$\left|e_n^{(k)}(t_n+\tau h_n)\right| \le \sum_{r=k}^{d} \frac{y^{(r)}(t_n)-u_{n-1}^{(r)}(t_n)}{(r-k)!}\tau^{r-k}h_n^{r-k} +$$

$$+h_n^{m+d+1-k}\left[\sum_{r=1}^{m}\frac{(r+d)!}{(r+d-k)!}|\beta_{n,r}|\tau^{r+d-k}+\frac{1}{(m+d-k)!}\int_0^\tau\left|y^{(m+d+1)}(t_n+sh_n)(\tau-s)^{m+d-k}\right|ds\right]$$

for all $\tau\in(0,1]$ and $k=1,2,\dots,d$, respectively, the inequality

$$\left|e_n^{(k)}(t_n+\tau h_n)\right| \le h_n^{m+d+1-k}\left[\sum_{r=k-d}^{m}\frac{(r+d)!}{(r+d-k)!}|\beta_{n,r}|\tau^{r+d-k}+\right.$$

$$\left.+\frac{1}{(m+d-k)!}\int_0^\tau\left|y^{(m+d+1)}(t_n+sh_n)(\tau-s)^{m+d-k}\right|ds\right]$$

for all $\tau\in(0,1]$ and $k=d+1,d+2,\dots,m+d$.

Since $\|\beta_n\|_1\le B_n$, $\left|y^{(m+d+1)}(t)\right|\le M$ for all $t\in\sigma_n$ and together with (2.24) it follows that there exists the constants $C_{n,k}, k=1,2,\dots,m+d$ such that:

(2.27) $$\left|e_n^{(k)}(t)\right| \le C_{n,k}h^{m+d+1-k},\quad t\in\sigma_n.$$

So, evaluations (2.26) and (2.27) end the proof of the theorem.

*Remark* 2.3. (i) If we take $d=-1$ and $m\ge1$, the theorem is reduced to the convergence theorem given by H. Brunner and P. J. van der Houwen in [2].

(ii) If we take $d=n, n\in\mathbb{N}$ and $m=1$, the above algorithm is identical with the algorithm presented by M. Micula and G. Micula in [5] and Theorem 2.2 is equivalent with the theorem obtained by M. Lubor in [4].

## 3. LOCAL SUPERCONVERGENCE

The notion of local superconvergence is used when on a set of interior points $Z_N$ (or $\overline{Z}_N$) the approximate solution has a convergence order greater than the global convergence order. Since in many practical problems we are interesed



only in the approximation of the solution on these points, it is very important to obtain results concerning local superconvergence.

In the following, to obtain some simpler proofs we shall consider instead of the nonlinear Volterra integral equation of the second kind (1.1) a linear integral equation of the following form:

(3.1) $$y(t)=f(t)+\int_0^t K(t,s)y(s)ds,\quad t\in[0,T]=:I,$$

in which the given functions $f$ and $K$, defined on $I$, respectively on $S$, are supposed to be sufficiently smooth. The local superconvergence results obtained for equation (3.1) are also valid for the integral equation (1.1) (see [2]).

From Theorem 2.2 we notice that the only conditions imposed on the collocation parameters are that they must be distinct and they must belong to (0,1]. The local superconvergence on $\overline{Z}_N$ is closely connected with the choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ (see [1], [2], [3]) and with the relation between their number and the number of the coefficients of the approximate solution determined from the smooth conditions.

We will give the following theorem concerning the aspects presented above:

THEOREM 3.1. *If* $m\ge d+2$ *and* $u\in S_{m+d}^{(d)}(Z_N)$ *is the approximate solution defined by* (2.1) *and* (2.5), *the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$, *with* $0<c_1<\dots<c_m=1$ *are chosen such that*:

(3.2) $$J_k:=\int_0^1 s^k\prod_{j=1}^{m}(s-c_j)ds=0,\ \text{for } k=0,1,\dots,p-1;$$

$$J_p\ne0,\quad\text{where}\quad d+1\le p\le m,$$

*and if f and K have continuous derivatives of order m+p, then the following estimation hold*:

(3.3) $$\max_{t_n\in Z_n}|y(t_n)-u(t_n)|=0(h^{m+p}),\ \text{for } h\to0 \text{ and } Nh\le\gamma T,$$

*where y is the solution of linear Volterra integral equation* (3.1).

*Remark* 3.2. It is known that the orthogonality conditions (3.2) imply that the $m$-point interpolator quadrature formula based on the abscissas $\{c_j\}_{j=\overline{1,m}}$ has degree of precision $m+p$. Since this degree of precision cannot exceed the value $2m-1$, we always have $p\le m-1$.

*Proof.* The collocation equation for $u \in S_{m+d}^{(d)}(Z_n)$, which holds only on the collocation points $X(N)$, can be written in the form

$$u(t)=f(t)+\int_0^t K(t,s)u(s)ds-\delta(t),\ t\in I, \tag{3.4}$$

where $\delta$ denotes a suitable function, subsequently called the defect function, vanishing on $X(N)$. This function is smooth on each subinterval $\sigma_n$, with the degree of smoothness given by that of $f$ and $K$. Moreover, due to the global convergence of $u$, $\delta(t)$ tends to zero, uniformly on $I$, as $h\to 0$.

Relations (3.4) and (3.1) yield to a second-kind integral equation for the error function

$$e(t)=\delta(t)+\int_0^t K(t,s)e(s)ds,\ t\in I, \tag{3.5}$$

whose solution is given by:

$$e(t)=\delta(t)+\int_0^t R(t,s)\delta(s)ds,\ t\in I, \tag{3.6}$$

where $R(t,s)$ denotes the resolvent kernel for $K(t,s)$. Substituting $t=t_n$ in (3.6), $n=1,2,...,N$ it follows

$$\begin{aligned} e(t_n) &= \delta(t_n)+\int_0^{t_n} R(t_n,s)\delta(s)ds= \\ &= \delta(t_n)+\sum_{i=0}^{n-1} h_i\int_0^1 R(t_n,t_i+\tau h_i)\delta(t_i+\tau h_i)d\tau. \end{aligned} \tag{3.7}$$

If each integral equation from equation (3.7) is approximation with an intepolator $m$-point quadrature formula based on the abscissas $\{t_{i,l}\}_{l=1,m}$ for all $i=0,1,...,n-1$ we are led to:

$$e(t_n)=\delta(t_n)+\sum_{i=0}^{n-1} h_i\left(\sum_{l=1}^{m} W_l R(t_n,t_{i,l})\delta(t_{i,l})+E_{n,i}\right)=\sum_{i=0}^{n-1} h_i E_{n,i}, \tag{3.8}$$

where $E_{n,i}$, represents the error of the interpolator quadrature formula used, and $W_l, l=1,2,...,m$ are the weight of these formulas:

By hypothesis (3.2) we get

$$|E_{n,i}|\le C_i h^{m+p},\qquad i=0,1,\dots,N-1, \tag{3.9}$$

and taking into account (3.8) and (3.9) it follows the estimation which proves the theorem

$$|e(t_n)|\le Nh\sup\{|E_{n,i}|: 0\le i\le n-1\le N-1\}=C\cdot h^{m+p},$$



COROLLARY 3.3. (i) *If the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the zeros of* $P_{m-1}(2s-1)-P_m(2s-1)$ (i.e., *the Radau II points for* (0,1]), *then in Theorem* 3.1 *we have* $p=m-1$ *and*

$$\max_{t_n\in Zn}|e(t_n)|=O(h^{2m-1})\quad (\text{as } h\searrow 0,\ Nh\le\gamma T). \tag{3.10}$$

(ii) *If the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are chosen such that the first* $m-1$ *of them are the zeros of* $P_{m-1}(2s-1)$ (i.e., *the Gauss points for* (0,1)), *and* $c_m=1$ *then approximating the integrals from equations (3.7) with interpolator* $m-1$*-point quadrature formula based on the abscissas* $\{t_{i,l}\}_{l=\overline{1,m}}$ *for all* $i=0,1,\dots,n-1$ *one obtains*

$$\max_{t_n\in Zn}|e(t_n)|=O(h^{2m-1})\quad (\text{as } h\searrow 0,\ Nh\le\gamma T). \tag{3.11}$$

*Remark* 3.4. To obtain the local superconvergence one must necessarily impose in Theorem 3.1 the condition that $c_m=1$. This condition implies that $t_n\in X(N)$ for all $n=1,2,...N$ and so $\delta(t_n)\equiv 0$. If $c_m<1$, then, in general, $\delta(t_n)\ne 0$; thus, to determine the order of $e(t_n)$ we require more precise information about the order of the defect function $\delta$ on the set $\overline{Z}_n$. In case $d=-1, m\ge 1$ the best order for the function $\delta$ is $\delta(t_n+\tau h_n)=O(h^m)$ for all $t_n+\tau h_n\notin X(N)$ (see [2], pp.256). This result proves the necessity of the condition $c_m=1$ to obtain the local superconvergence results.

## 4. NUMERICAL EXAMPLES

Consider the Volterra integral equations of the second kind

$$y(t)=e^t-\int_0^t e^{t-s}y(s)ds,\ t\in[0,1], \tag{4.1}$$

which has the exact solution $y(t)=1$ for all $t\in[0,1]$; and

$$y(t)=e^t-\int_0^t e^{t-s}\left(y(s)-e^{-y(s)}\right)ds,\ t\in[0,1], \tag{4.2}$$

which has the exact solution $y(t)=\ln(t+e)$ for all $t\in[0,1]$.

The exact solutions of these equations will be approximated by the exact collocation method. The integrals occurring in the collocation equation (2.5) are evaluated analytically if the method is used for solving integral equation (4.1). In the second case, when the method is used for solving integral equation (4.2) the integrals occurring in the collocation equation (2.5) are evaluated using the extended type precision from Pascal. The resulting nonlinear algebraic systems were solved by the Newton method.

We choose $m = 3$ and $d \in \{-1,0,1\}$; the collocation parameters are, respectively, the Radau II points $\left(c_1 = \left(4-\sqrt{6}\right)/10, c_2 = \left(4+\sqrt{6}\right)/10, c_3 = 1\right)$, and the Gauss points $\left(c_1 = \left(3-\sqrt{3}\right)/6, c_2 = \left(3+\sqrt{3}\right)/6\right)$ together with $c_3 = 1$.

Table 1 contains the values of:

$$\|e\|_\infty := \sup\{|e_n(t)| : t \in \sigma_n,\ n = 0,1,\dots,N-1\}$$

*Table 1*

Collocation at the Radau II points

| $d$ | $N(h)$ | $\|e\|_\infty$ | |
|---|---|---|---|
| | | Eq. 4.1 | Eq. 4.2 |
| -1 | 10 (0.1) | $2.43 \times 10^{-2}$ | $6.21 \times 10^{-2}$ |
| | 50 (0.02) | $8.15 \times 10^{-4}$ | $1.37 \times 10^{-3}$ |
| | 100 (0.01) | $3.72 \times 10^{-6}$ | $4.62 \times 10^{-5}$ |
| 0 | 10 (0.1) | $3.64 \times 10^{-2}$ | $7.32 \times 10^{-2}$ |
| | 50 (0.02) | $7.24 \times 10^{-4}$ | $2.81 \times 10^{-3}$ |
| | 100 (0.01) | $5.45 \times 10^{-6}$ | $6.53 \times 10^{-5}$ |
| 1 | 10 (0.1) | $5.51 \times 10^{-2}$ | $8.06 \times 10^{-2}$ |
| | 50 (0.02) | $9.27 \times 10^{-4}$ | $4.43 \times 10^{-3}$ |
| | 100 (0.01) | $2.03 \times 10^{-5}$ | $8.412 \times 10^{-5}$ |

*Table 2*

Collocation at the Gauss points

| $d$ | $N(h)$ | $\|e\|_\infty$ | |
|---|---|---|---|
| | | Eq. 4.1 | Eq. 4.2 |
| -1 | 10 (0.1) | $1.84 \times 10^{-2}$ | $3.12 \times 10^{-2}$ |
| | 50 (0.02) | $5.28 \times 10^{-3}$ | $1.89 \times 10^{-3}$ |
| | 100 (0.01) | $2.53 \times 10^{-4}$ | $5.24 \times 10^{-4}$ |
| 0 | 10 (0.1) | $4.71 \times 10^{-2}$ | $7.42 \times 10^{-2}$ |
| | 50 (0.02) | $7.63 \times 10^{-3}$ | $5.78 \times 10^{-3}$ |
| | 100 (0.01) | $3.27 \times 10^{-4}$ | $8.37 \times 10^{-4}$ |
| 1 | 10 (0.1) | $6.32 \times 10^{-2}$ | $1.03 \times 10^{-1}$ |
| | 50 (0.02) | $9.18 \times 10^{-3}$ | $2.41 \times 10^{-2}$ |
| | 100 (0.01) | $1.67 \times 10^{-3}$ | $1.83 \times 10^{-3}$ |

obtained by employing collocation at the Radau II point. These values are computed for every integral equation above. The results given in Table 2 are those obtained for the Gauss points.



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