REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 24, N^{os} 1–2, 1995, pp. 75–89

THE NUMERICAL TREATMENT OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND BY THE DISCRETIZED COLLOCATION METHOD

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1. INTRODUCTION

In [5] we have described an algorithm for numerical solution of nonlinear Volterra integral equation of the second kind:

$$(1.1)\qquad y(t) = f(t) + \int_0^t K(t,s,y(s))\,\mathrm{d}s,\quad t\in I:=[0,T],$$

in the space of polynomial spline functions of degree $m+d$ and continuity of class d, $S_{m+d}^{(d)}(Z_N)$ $(m\geq 1,\ d\geq -1)$.

Using the notation and the definitions given in [5], the exact collocation equation (2.3) from [5] can be written:

$$(1.2a)\qquad u(t_{n,j}) = f(t_{n,j}) + h_n\phi_{n,n}^{(j)}[u_n] + F_n(t_{n,j}),$$

where

$$(1.2b)\qquad F_n(t_{n,j}) := \sum_{i=0}^{n-1} h_i\phi_{n,i}^{(j)}[u_i]$$

denotes the lag term and $\phi_{n,i}^{(j)}[u_i]$, $i=\overline{1,n}$ denote the following integrals (see [1], [3])

$$(1.3)\qquad \phi_{n,i}^{(j)}[u_i] := \begin{cases} \int_0^1 K(t_{n,j},\, t_i+\tau h_i,\, u_i(t_i+\tau h_i))\,\mathrm{d}\tau, & \text{if } 0\leq i\leq n-1\\ \int_0^{c_j} K(t_{n,j},\, t_n+\tau h_n,\, u_n(t_n+\tau h_n))\,\mathrm{d}\tau, & \text{if } i=n\ (j=1,\dots,m). \end{cases}$$

From [7], we have that an element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined when we know the coefficients $\{a_{n,r}\}_{r=\overline{1,m}}$ for all $n = 0,1, \ldots, N\text{-}1$ (see (2.1) from [5]). Equation (1.2) represents, for each $n = 0,1, \ldots, N\text{-}1$ a recursive system which will give these coefficients.

In the case in which integrals (1.3) can be evaluated analytically the problem of determining the approximative solution $u \in S_{m+d}^{(d)}(Z_N)$ and the convergence and local superconvergence properties of this solution had already been studied in [5].

In this paper we will study the case in which integrals (1.3) occurring in the exact collocation equations cannot be evaluated analytically.

2. THE DISCRETIZED COLLOCATION EQUATION

In most applications integrals (1.3) occurring in the exact collocation equations (1.2) cannot be evaluated analytically, and one is forced to resort to employing suitable quadrature formulas for their approximation. In the following we suppose that these integrals are approximated by quadrature formulas of the form (see [1], [2], [3], [4]):

(2.1a) $$\hat{\phi}_{n,i}^{(j)}[u_i] := \sum_{l=1}^{\mu_1} w_l K\big(t_{n,j}, t_i + d_l h_i, u_i(t_i + d_l h_i)\big),$$

and

(2.1b) $$\hat{\phi}_{n,n}^{(j)}[u_n] := \sum_{l=1}^{\mu_0} w_{j,l} K\big(t_{n,j}, t_n + d_{j,l} h_n, u_i(t_n + d_{j,l} h_n)\big),$$

where μ_0 and μ_1 are two given positive integers. These quadrature formulas are usually interpolatory ones, with the parameters $\{d_l\}$ and $\{d_{j,l}\}$ satisfying, respectively:

$$0 \le d_1 < \ldots < d_{\mu_1} \le 1 \quad \text{and} \quad 0 \le d_{j,1} < \ldots < d_{j,\mu_0} \le c_j \ (j = 1,\ldots,m).$$

The quadrature weights are then given by:

$$w_l := \int_0^1 \prod_{\substack{r=1 \\ r\neq l}}^{\mu_1} (s - d_r)/(d_l - d_r)\,ds, \quad l = 1,\ldots,\mu_1$$

and

$$w_{j,l} := \int_0^{c_j} \prod_{\substack{r=1 \\ r\neq l}}^{\mu_0} \big(s - d_{j,r}\big)/\big(d_{j,l} - d_{j,r}\big)\,ds, \quad l = 1,\ldots,\mu_0,\ j = 1,\ldots,m.$$

and the corresponding error terms are defined by:

(2.2) $$E_{n,i}^{(j)}[u_i] = \phi_{n,i}^{(j)}[u_i] - \hat{\phi}_{n,i}^{(j)}[u_i], \quad j = 1,\ldots,m \ (i = 0,\ldots,n),$$

with $\phi_{n,j}^{(j)}[u_i]$ and $\hat{\phi}_{n,i}^{(j)}[u_i]$ given by (1.3) and (2.1).

We now use the quadrature formulas to obtain the fully discretized version of the exact collocation equations (1.2). Since the quadrature error terms will be disregarded, we generate an approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ which has for all n=0, 1, ..., N-1 and for all $t \in \sigma_n$ the following form:

(2.3) $$\hat{u}(t) = \hat{u}_n(t) = \sum_{r=0}^{d} \frac{\hat{u}_{n-1}^{(r)}(t_n)}{r!}(t - t_n)^r + \sum_{r=1}^{m} \hat{a}_{n,r}(t - t_n)^{d+r},$$

with

$$\hat{u}^{(r)}(0) = y^{(r)}(0), \quad r = 0,1,\ldots,d,$$

and which is defined by:

(2.4a) $$\hat{u}_n(t_{n,j}) = f(t_{n,j}) + h_n \hat{\phi}_{n,n}^{(j)}[\hat{u}_n] + \hat{F}_n(t_{n,j}),$$

where

(2.4b) $$\hat{F}_n(t_{n,j}) = \sum_{i=0}^{n-1} h_i \hat{\phi}_{n,i}^{(j)}[\hat{u}_i]$$

denotes the approximation to the lag term.

One can observe that the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$, given by the fully discretized collocation equations (2.4) will, in general, be different from the approximation $u \in S_{m+d}^{(d)}(Z_N)$ given by the exact collocation equations (1.2). Denote by $\hat{e}^{(k)} = y^{(k)} - \hat{u}^{(k)}$, $k = 0,\cdots,m+d$ the approximation error of the solution y and of its derivates up to k-th order by the approximate solution $\hat{u}$ and respectively by its derivates. Also denote by $\hat{e}_n^{(k)}$ to the restriction of $\hat{e}^{(k)}$ to the subinterval σ_n for all n=0,1,..., N-1; $\hat{e}_n^{(k)}(t) := \hat{e}^{(k)}(t)\big|_{t\in\sigma_n}$. The order of $\hat{e}^{(k)}$, k=0,1, ..., $m+d$ will depend on the choice of quadrature formulas (2.1), as described in the following theorem:

THEOREM 2.1. *If the nonlinear Volterra integral equation of the second kind* (1.1) $f \in C^{m+d+1}(I)$ *and* $K \in C^{m+d+1}(S\times\mathbb{R})$ *and if the quadrature formulas* (2.1) *and* (2.2) *satisfy:*

(2.5) $$\int_0^1 \varphi(t_i + \tau h_i)\,d\tau - \sum_{l=1}^{\mu_1} w_l \varphi(t_i + d_l h_i) = O\big(h_i^{s_1}\big), \quad i = 0,\ldots,n-1,$$

and

(2.6) $$\int_0^{c_j} \varphi(t_n + \tau h_n)\, d\tau - \sum_{l=1}^{\mu_0} w_{j,l}\varphi(t_n + d_{j,l}h_n) = O(h_n^{s_0}),\ j = 1,2,\ldots,m,$$

whenever the integrand is a sufficiently smooth function, then for any choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ *with* $0<c_0<\ldots<c_m\le 1$ *and for all quasi-uniform meshes with sufficiently small* $h>0$, *we have:*

(2.7) $$\left\|\hat{e}^{(k)}\right\|_\infty \le \hat{C}_k h^{s-k},\ k = 0,1,\ldots,s,$$

with $s := \min\{m+d+1,\ s_0+1,\ s_1\}$ *and* $\hat{C}_k$ *are finite constants independent of* h.

Proof. We shall prove it by induction using the same technique as in the proof of Theorem 2.2 from [5].

First, we develop the exact solution y *in* $\sigma_0 = [0, t_0]$ in Taylor series in the neighborhood of the origin, and we obtain for all $\tau\in[0,1]$, that:

(2.8) $$y(\tau h_0) = \sum_{r=0}^{m+d} \frac{y^{(r)}(0)}{r!}\tau^r h_0^r + R_0(\tau)\cdot h_0^{m+d+1},$$

where

$$R_0(\tau) := \frac{y^{(m+d+1)}(\xi_0)}{(m+d+1)!}\tau^{m+d+1},\ 0<\xi_0<\tau h_0.$$

So, by (2.3) with n=0, we have for all $\tau\in[0,1]$

(2.9a) $$\hat{e}(\tau h_0) = y(\tau h_0) - \hat{u}_0(\tau h_0) = h_0^{m+d+1}\left[\sum_{r=1}^{m}\hat{\beta}_{0,r}\tau^{d+r} + R_0(\tau)\right],$$

where

(2.9b) $$h_0^{m+1}\beta_{0,r} := \left(\frac{y^{(d+r)}(0)}{(d+r)!} - \hat{a}_{0,r}\right)h_0^r,\ r = 1,2,\ldots,m.$$

Since y is the exact solution of integral equation (1.1), then for all j=1,2,...,m it fulfills the equation

$$y(c_j h_0) = f(c_j h_0) + h_0\phi_{0,0}^{(j)}[y],\ j = 1,2,\ldots,m,$$

which together with (2.1b) and (2.2) (n=0) gives

(2.10) $$y(c_j h_0) = f(c_j h_0) + h_0\hat{\phi}_{0,0}^{(j)}[y] + h_0 E_{0,0}^{(j)}[y].$$

By (2.4), (2.9) and (2.10) we obtain the system:

(2.11) $$\left(V - h_0\hat{D}_0\right)\hat{\beta}_0 = \hat{q}_0 - R_0 + h_0^{-m-d}r_0,$$

where

$$\hat{\beta}_0 := \left(\hat{\beta}_{0,r}\right)_{r=\overline{1,m}},$$

$$R_0 := \left(R_0(c_j)\right)_{j=\overline{1,m}},$$

$$V := \left(c_j^{d+r}\right)_{j,r=\overline{1,m}},$$

$$\hat{D}_0 := \left(\sum_{l=1}^{\mu_0} w_{j,l}\frac{\partial K}{\partial y}\left(c_j h_0, d_{j,l}h_0, \hat{Z}(d_{j,l}h_0)\right)d_{j,l}^{d+r}\right)_{j,r=\overline{1,m}},$$

with $\hat{Z}(t)\in[y(t),\hat{u}(t)]$, for all $t\in\sigma_0$,

$$r_0 = \left(E_{0,0}^{(j)}[y]\right)_{j,r=\overline{1,m}},$$

$$\hat{q}_0 := \left(\hat{q}_{0,j}\right)_{j=\overline{1,m}},$$

with

$$\hat{q}_{0,j} := h_0\sum_{l=1}^{\mu_0} w_{j,l}\frac{\partial K}{\partial y}\left(c_j h_0, d_{j,l}h_0, \hat{Z}(d_{j,l}h_0)\right)R_0(d_{j,l}).$$

By (2.11) and (2.5) it follows that (see [5]) there exist the finite constants Q_0 and P_0 such that:

(2.12) $$\left\|\hat{\beta}_0\right\|_1 := \sum_{l=1}^{m}\left|\hat{\beta}_{0,l}\right| \le Q_0 + P_0 h^{-m-d+s_0},$$

the estimation which together with (2.9a) prove that:

(2.13) $$\sup\left\{|\hat{e}(t)| : t\in\sigma_0\right\} \le \hat{C}_0 h^s.$$

The estimations $\left|\hat{e}^{(k)}(t)\right| \le \hat{C}_{0,k}h^{s-k}$ (k=0,1,...,s) easily follow by (2.13), (2.8), (2.9a).

Suppose now that if for all j=0,1,...,n-1

(2.14) $$\left|\hat{e}_j^{(k)}(t)\right| \le \hat{C}_{j,k}h^{s-k},\ t\in\sigma_j,\ k = 0,1,\ldots,s$$

hold we shall prove that (2.14) holds for j=n. Therefore we develop the exact solution y in the interval σ_n in Taylor series

$$y(t_n+\tau h_n)=\sum_{r=1}^{m}\frac{y^{(r)}(t_n)}{r!}\tau^r h_n^r+h_n^{m+d+1}R_n(\tau), \tag{2.15}$$

where

$$R_n(\tau)=\frac{1}{(m+d)!}\int_0^{\tau}y^{(m+d+1)}(t_n+sh_n)(\tau-s)^{m+d}\,\mathrm{d}s,$$

for all $\tau\in(0,1]$.

So, by (2.3) and (2.15) we have

$$\hat{e}_n(t_n+\tau h_n)= \tag{2.16a}$$

$$=\sum_{r=0}^{d}\frac{y^{(r)}(t_n)-\hat{u}_{n-1}^{(r)}(t_n)}{r!}\tau^r h_n^r+h_n^{m+d+1}\left[\sum_{r=1}^{m}\hat{\beta}_{n,r}\tau^{r+d}+R_n(\tau)\right],$$

where

$$h_n^{m+1}\hat{\beta}_{n,r}=\left(\frac{y^{(d+r)}}{(d+r)!}-\hat{a}_{n,r}\right)h_n^r. \tag{2.16b}$$

Taking into account that y satisfies integral equation (1.1) and using the quadrature formulas (2.1) and (2.2) we have that:

$$y(t_{n,j})=f(t_{n,j})+\sum_{i=0}^{n-1}h_i\hat{\phi}_{n,i}^{(j)}[y]+\sum_{i=0}^{n-1}h_iE_{n,i}^{(j)}[y]+ \tag{2.17}$$

$$+h_i\hat{\phi}_{n,n}^{(j)}[y]+h_nE_{n,n}^{(j)}[y],\ j=1,2,\ldots,n.$$

From (2.4), (2.16) and (2.17) we obtain the following system:

$$(\mathrm{V}-h_n\cdot\hat{D}_{n,n})\hat{\beta}_n=\sum_{i=0}^{n-1}h_i\left[\left(\frac{h_i}{h_n}\right)^{m+d+1}\hat{D}_{n,i}\hat{\beta}_i+\frac{1}{h_n^{m+d+1}}\hat{F}_{n,i}\cdot\hat{E}_i\right]+ \tag{2.18}$$

$$+\frac{1}{h_n^{m+d+1}}\cdot\left(h_n\hat{F}_{n,n}-W\right)\hat{E}_n+\hat{q}_n+\frac{1}{h_n^{m+d+1}}\left[h_nr_{n,n}+\sum_{i=0}^{n-1}h_ir_{n,i}\right],$$

where

$$V:=\left(c_j^{r+d}\right)_{j,r=\overline{1,m}},$$

$$W:=\left(c_j^{r}\right)_{j,r=\overline{1,m}},$$

$$\hat{E}_i:=\left(h_i^r\frac{y^{(r)}(t_i)-\hat{u}_{i-1}^{(r)}}{r!}\right)_{r=\overline{0,d}},\ (i=0,1,\ldots,n),$$

$$\hat{D}_{n,i}:=\begin{cases}\left(\sum_{l=1}^{\mu_1}\frac{\partial K}{\partial y}\left(t_{n,j},\,t_i+d_lh_i,\,Z_i(t_i+d_lh_i)\right)d_l^{d+r}\right)_{j,r=\overline{1,m}}, & \text{if } 0\le i\le n-1,\\ \left(\sum_{l=1}^{\mu_0}\frac{\partial K}{\partial y}\left(t_{n,j},\,t_n+d_{j,l}h_n,\,Z_n(t_n+d_{j,l}h_n)\right)d_{j,l}^{d+r}\right)_{j,r=\overline{1,m}}, & \text{if } i=n,\end{cases}$$

with $Z_i(t)\in[y(t),\hat{u}_i(t)]$, for all $t\in\sigma_i$ and $i=0,1,\ldots,N-1$,

$$\hat{F}_{n,i}:=\begin{cases}\left(\sum_{l=1}^{\mu_1}\frac{\partial K}{\partial y}\left(t_{n,j},\,t_i+d_lh_i,\,Z_i(t_i+d_lh_i)\right)d_l^{r}\right)_{j=\overline{1,m},\,r=\overline{0,d}}, & \text{if } 0\le i\le n-1,\\ \left(\sum_{l=1}^{\mu_0}\frac{\partial K}{\partial y}\left(t_{n,j},\,t_n+d_{j,l}h_n,\,Z_n(t_n+d_{j,l}h_n)\right)d_{j,l}^{r}\right)_{j=\overline{1,m},\,r=\overline{0,d}}, & \text{if } i=n,\end{cases}$$

$$\hat{\beta}_i:=\left(\hat{\beta}_{i,r}\right)_{r=\overline{1,m}},\ (i=0,1,\ldots,n),$$

$$\hat{q}_n:=\left(\hat{q}_{n,j}\right)_{j=\overline{1,m}},$$

with

$$\hat{q}_{n,j}:=-R_n(c_j)+h_n\sum_{l=1}^{\mu_0}w_{j,l}\frac{\partial k}{\partial y}\left(t_{n,j},\,t_n+d_{j,l}h_n,\,Z_n(t_n+d_{j,l}h_n)\right)R_n(d_{j,l})+$$

$$+\sum_{i=0}^{n-1}h_i\left(\frac{h_i}{h_n}\right)^{m+d+1}\sum_{l=1}^{\mu_1}\frac{\partial k}{\partial y}\left(t_{n,j},\,t_i+d_lh_i,\,Z_i(t_i+d_lh_i)\right)R_i(d_l),$$

and

$$r_{n,i}:=\left(E_{n,i}^{(j)}[y]\right)_{j=\overline{1,m}},\ i=0,1,\ldots,n.$$

By (2.5), (2.6), (2.14) and (2.18) it follows that (see [5]) there exist the finite constants M_i, i=1,2,...,5 such that:

$$\left\|\hat{\beta}_n\right\|_1\le hM_1\sum_{i=0}^{n-1}\left\|\hat{\beta}_i\right\|_1+\left(M_2h^s+M_3h^{s_0+1}+M_4h^{s_1}\right)/h^{m+d+1}+M_5. \tag{2.19}$$

This represents a discrete Gronwall inequality for $\left\|\hat{\beta}_n\right\|_1$,

$$(2.20)\qquad \left\|\hat{\beta}_n\right\|_1 \le hM_1\sum_{i=0}^{n-1}\left\|\hat{\beta}_i\right\|_1 + \frac{h^s}{h^{m+d+1}}M_6 + M_5,\quad n = 0,\dots,N-1,$$

where $M_6 := M_2 + M_3\, h^{s_0+1-s} + M_4\, h^{s_1-s}$. Thus, by Corollary 1.52 from [3] it follows that there exist finite constants Q_n and P_n such that:

$$(2.21)\qquad \left\|\hat{\beta}_n\right\|_1 \le Q_n + P_n h^{s-m-d-1}$$

for all n=0,1,...,N-1 with $Nh \le \gamma T$ and h>0 sufficiently small.

Now by (2.21), (2.16) and (2.14) we obtain that:

$$(2.22)\qquad \left|\hat{e}_n(t_n + \tau h_n)\right| \le \hat{C}_n h^s,$$

for all $\tau \in (0,1]$, where $\hat{C}_n$ are the positive constants.

Deriving relations (2.3) and (2.15) k times (k=1,2,...,s) and using (2.14) and (2.21) we easily obtain that:

$$(2.23)\qquad \left|\hat{e}_n^{(k)}(t_n + \tau h_n)\right| \le \hat{C}_{n,k} h^{s-k}$$

for all $\tau \in (0,1]$ with $\hat{C}_{n,k}$ the positive constants.

COROLARRY 2.2. *Let the assumptions of* Theorem 2.1 *hold. If the quadrature formulas* (2.1) *are of interpolatory type, with* $\mu_0 = \mu_1 = m+d+1$, *then the approximation* $\hat{u} \in S_{m+d}^{(d)}(Z_n)$ *defined by the discretized collocation equation* (2.4) *leads to an error* $\hat{e}(t)$ *satisfying*

$$(2.24)\qquad \|\hat{e}\|_\infty = O\left(h^{m+d+1}\right),\ (\text{as } h\searrow 0 \text{ and } Nh \le \gamma T)$$

for every choice of the collocation parameters $\{c_j\}$ *with* $0<c_1<\dots<c_m\le 1$.

Proof. It is know that an m+d+1- point interpolatory quadrature formula is characterized, in the terminology of Theorem 2.1, by $\min(s_0, s_1) \ge m+d+1$. Hence, we have $s = m+d+1$.

Now, we consider the approximation $u \in S_{m+d}^{(d)}(Z_n)$ defined by the exact collocation equation (1.5) and denote by $\varepsilon := u - \hat{u}$ the difference between the approximation u and the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_n)$ defined by the discretized

collocation equation (2.4), if the assumptions of Theorem 2.1 hold, then one can easily prove that the order of this difference is $s = \min(m+d+1,\ s_0+1,\ s_1)$, thus:

$$\|\varepsilon\|_\infty := \|u-\hat{u}\|_\infty \le \|u-y\|_\infty + \|y-\hat{u}\|_\infty \le Ch^{m+d+1} + \hat{C}h^s \le \hat{Q}h^s,$$

where we used the results of Theorem 2.1 and Theorem 2.2 from [5]. But we can prove that the order of ε will depend only on the choice of the quadrature formulas (2.1), as described in the following theorem.

THEOREM 2.3. *Let the assumptions of* Theorem 2.1 *hold, then there exists a finite constant Q such that* $\varepsilon := u - \hat{u}$ *satisfies*

$$(2.25)\qquad \|\varepsilon\|_\infty \le Qh^{s'},\ \text{ with } s' = \min(s_0+1, s_1),$$

for all quasi-uniform meshes with sufficiently small $h > 0$.

Proof. Let $\varepsilon_n(t)$ denote the restriction of $\varepsilon(t)$ to the subinterval σ_n. Subtracting (2.4) from (1.2) and using (2.2) we obtain the following recurrence relation:

$$(2.26)\quad \varepsilon_n(t_{n,j}) = h_n\sum_{l=1}^{\mu_0} w_{j,l}\frac{\partial K}{\partial y}\left(t_{n,j}, t_n + d_{j,l}h_n, Z_n(t_n + d_{j,l}h_n)\right)\varepsilon_n(t_n + d_{j,l}h_n) +$$
$$+h_n E_{n,n}^{(j)}[u_n] + \sum_{i=0}^{n-1} h_i \sum_{l=1}^{\mu_1} w_l \frac{\partial K}{\partial y}\left(t_{n,j}, t_i + d_l h_i, Z_i(t_i + d_l h_i)\right)\varepsilon_i(t_i + d_l h_i) +$$
$$+\sum_{i=0}^{n-1} h_i E_{n,i}^{(j)}[u_i],\quad j = 1,\dots,m,$$

where $Z_n(t) \in [u_n(t), \hat{u}_n(t)]$, for all $t \in \sigma_n$ (n=0,1,...,N-1).

By (2.1) from [5] and (2.3) we have that for all $\tau \in (0,1]$

$$(2.27)\ \varepsilon_n(t_n + \tau h_n) = \sum_{r=0}^{d}\frac{\varepsilon_{n-1}^{(r)}(t_n)}{r!}(\tau h_n)^r + \sum_{r=1}^{m}\left(a_{n,r} - \hat{a}_{n,r}\right)(\tau h_n)^{d+r},\ n = 0,\dots,N-1.$$

If we denote by $\eta_n := \left(\eta_{n,1}, \eta_{n,2},\dots,\eta_{n,m}\right)$ with $\eta_{n,r} := \left(a_{n,r} - \hat{a}_{n,r}\right)h_n^{d+r}$ using the notation from the proof of Theorem 2.1 then by (2.26) and (2.27) we obtain the following relation:

$$(2.28)\qquad \left(V - h_n\cdot\hat{D}_{n,n}\right)\eta_n = \sum_{i=0}^{n-1} h_i\left[\hat{D}_{n,i}\eta_i + \hat{F}_{n,i}\cdot\mathscr{E}_i\right] +$$
$$+\left(h_n\hat{F}_{n,n} - W\right)\mathscr{E}_n + h_n r_{n,n} + \sum_{i=0}^{n-1} h_i r_{n,i},$$

where

$$\mathscr{E}_i = \left(\frac{\varepsilon_{i-1}^{(j)}(t_i) h_i^r}{r!} \right)_{r=0,d}, \quad i = 0,1,\ldots,n.$$

This equation is analogue to (2.18). Now using the same technique as in the proof of Theorem 2.1 or of Theorem 2.2 from [5] one can prove that $\|\eta_n\|_1 = O(h^{s'})$ and $\|\mathscr{E}_n\|_1 = O(h^{s'})$ and thus by (2.27) it follows that there exists

$$\|\varepsilon\|_\infty \le Q h^{s'}, \text{ with } s' = \min\{s_0 + 1, s\}.$$

a finite constant Q such that:

Remark 2.4. (i) Theorem 2.3 and Theorem 2.2 from [5] imply Theorem 2.1, because we can write $\|\hat{e}\|_\infty = \|y - \hat{u}\|_\infty \le \|u - y\|_\infty + \|u - \hat{u}\|_\infty = \|y - u\|_\infty + \|\varepsilon\|_\infty$.

(ii) If we take $d = -1$ and $m \ge 1$ the above theorem and corollary are reduced to the theorems given by H. Brunner and P.J. van der Houwen in [3], pp. 260–262.

In the numerical applications it is very important that the convergence order of the methods used to be the highest possible. From Theorem 2.1 it follows that the highest convergence order, in the exact collocation method for m and d fixed, is $s = m+d+1$, which is obtained when the quadrature formulas used are such that s_0+1 and s_1 are greater than $m+d+1$. Also, to reduce the volume of computations it is useful to employ the simplest possible quadrature formulas and which have highest degree of precision. For instance, if we consider $\mu = \mu_0 = \mu_1$ and $d_{j,l} = d_j \cdot d_l$ then:

(i) if $2\mu \ge m+d+1$, $\{d_l\}_{l=\overline{1,\mu}}$ are the Gauss points for (0,1) and quadrature formulas (2.1) are of the Gauss quadrature formulas, then we have $s = m+d+1$;

(ii) if $2\mu \ge m+d+2$, $\{d_l\}_{l=\overline{1,\mu}}$ are the Radau II points for (0,1] and quadrature formulas (2.1) are of the Radan quadrature formulas, then we have $s = m+d+1$.

In many papers (see [1], [2], [3], [4]) the quadrature formulas used have $\mu_0 = \mu_1 = m$, $d_j = c_j$ and $d_{j,l} = c_j c_l$ $(j,l = \overline{1,m})$. The possibility of employing some quadrature formulas of this type in our method would lead us to some simplifications. These simplifications are useful when they do not spoil the convergence order given by Theorem 1.1, namely $s = m+d+1$ in Theorem 2.1. An answer to this problem is given in the following corollary.

COROLLARY 2.5. *If in nonlinear Volterra integral equation of the second kind* (1.1), $f \in C^{m+d+1}(I)$ and $K \in C^{m+d+1}(S \times \mathbb{R})$ *and if* $m \ge d+1$, *then there exists the set of collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *such that for the approximation* $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ *given by the discrete collocation equations* (2.4) *in which* $\mu_0 = \mu_1 = m$, $d_j = c_j$ *and* $d_{j,l} = c_j c_l$ *we have*

$$(2.29) \qquad \|\hat{e}\|_\infty := \|y - \hat{u}\|_\infty = O\left(h^{m+d+1}\right).$$

Proof. If $\mu = \mu_0 = \mu_1 = m$ and $m \ge d+1$, then it follows that $2\mu \ge m+d+1$ and by the above remark that there exist the m Gauss points for (0,1) such that (2.29) holds. If $\mu = m \ge d+2$ then it follows that $2\mu \ge m+d+2$ and there exist the m Radau II points for (0,1] such that (2.29) holds.

If the kernel $K(t,s,y)$ can be smooth extended to $S' \times \mathbb{R}$, where $S' = \{(t,s) : 0 \le s \le t + \delta\} \cap I \times I$, for some $\delta > 0$ then the integrals $\phi_{n,n}^{(j)}[u_n]$ in (1.2) may be approximated choosing $\mu_0 = \mu_1$, $d_{j,l} = d_l$,

$$(2.30) \qquad \tilde{\phi}_{n,n}^{(j)}[u_n] := \sum_{l=1}^{\mu_1} \tilde{w}_{j,l} K(t_{n,j}, t_n + d_l h_n, u_n(t_n + d_l h_n)),$$

with

$$\tilde{w}_{j,l} := \int_0^{c_j} \prod_{\substack{r=1 \\ r \ne l}}^{\mu_1} (s - d_r)/(d_l - d_r) \mathrm{d}s,$$

and the corresponding error terms are defined by:

$$(2.31) \qquad \tilde{E}_{n,n}^{(j)}[u_n] = \phi_{n,n}^{(j)}[u_n] - \tilde{\phi}_{n,n}^{(j)}[u_n], \quad j = 1,\ldots,m.$$

Using quadrature formula (2.30) in the discretized collocation equation we have the following equation:

$$(2.32) \qquad \tilde{u}(t_{n,j}) = f(t_{n,j}) + h_n \tilde{\phi}_{n,n}^{(j)}[\tilde{u}_n] + \sum_{i=0}^{n-1} h_i \tilde{\phi}_{n,i}^{(j)}[\tilde{u}_i], \; j = 1,\ldots,m \; (n = 0,\ldots,N-1),$$

where the approximations $\tilde{u} \in S_{m+d}^{(d)}(Z_N)$ have forms analogous to (2.3) for all $n = 0,1,\ldots,N-1$ and for all $t \in \sigma_n$.

If we denote by $\tilde{e} := y - \tilde{u}$ the approximation error of the solution y by the approximate solution $\tilde{u}$ and by $\varepsilon := u - \tilde{u}$ the difference of the approximation u by the approximation $\tilde{u}$, then repeating the above reasoning one can prove the following theorem:

THEOREM 2.6. *Suppose that the given functions* $f \in C^{m+d+1}(I)$ *and* $K \in C^{m+d+1}(S' \times \mathbb{R})$, *then the following assertions are true:*

a) *if the quadrature formulas* (2.1a) *and* (2.30) *have the convergence order* s_1, *then for any choice of the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *with* $0 < c_0 < \ldots < c_m \le 1$ *and for all quasi-uniform meshes with sufficiently small* $h > 0$, *we have:*

$$\|\tilde{e}\|_\infty := \|y-\tilde{u}\|_\infty = O(h^s)$$

and

$$\|\tilde{\varepsilon}\|_\infty := \|u-\tilde{u}\|_\infty = O(h^{s_1}),$$

with $s := \{m+d+1, s_1\}$;

b) if the quadrature formulas (2.1a) *and* (2.30) *are of interpolatory type, with* $\mu_1 = m+d+1$, *then*

$$\|\tilde{e}\|_\infty := \|y-\tilde{u}\|_\infty = O(h^{m+d+1});$$

c) if $m \geq d+1$, *then there exists the set of collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *such that for the approximation* $\tilde{u} \in S^{(d)}_{m+d}(Z_N)$ *given by the discrete collocation equations* (2.32) *in which* $\mu_1 = m$, $d_j = c_j$ *and* $d_{j,l} = c_j c_l$, *we have*

$$\|\tilde{e}\|_\infty := \|y-\tilde{u}\|_\infty = O(h^{m+d+1}).$$

Remark 2.7: (i) The results of the above theorem for $d = -1$ and $m \geq 1$ is reduced to the results given in [3].

(ii) Other possibilities for discretization of $\phi^{(j)}_{n,n}[u_n]$ can be found in [3].

3. LOCAL SUPERCONVERGENCE

We now deal with the question of the attainable order of superconvergence (on $\bar{Z}_N$) in approximations $\hat{u} \in S^{(d)}_{m+d}(Z_N)$ defined by the fully discretized collocation equation (2.4) and, respectively, in approximations $\tilde{u} \in S^{(d)}_{m+d}(Z_N)$ defined by (2.32). As in [5], we state the results for the linear integral equation

$$(3.1)\qquad y(t) = f(t) + \int_0^t K(t,s)y(s)ds\ ,\quad t \in I := [0,T]\ ,$$

the modification for the general case being straightforward. It is again assumed that the underlying mesh sequence is quasi-uniform. We have the following theorem:

THEOREM 3.1. *If* $m \geq d+2$ *and* $\hat{u}, \tilde{u} \in S^{(d)}_{m+d}(Z_N)$, *denote, respectively, the collocations approximations determined by* (2.4) *and* (2.32), *the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$, *with* $0 < c_1 < \ldots < c_m = 1$ *are chosen such that:*

$$(3.2)\qquad J_k := \int_0^1 s^k \prod_{j=1}^m (s-c_j)ds = 0, \text{ for } k = 0,1,\ldots,p-1,$$

$$J_p \neq 0 \qquad , \text{ where } d+1 \leq p \leq m,$$

and iff and K have continuous derivates of sufficiently high order on their respective domains, then the following estimations hold:

$$(3.3)\qquad \max_{t_n \in \bar{Z}_N} |y(t_n) - \hat{u}(t_n)| = O(h^\alpha)$$

and

$$(3.4)\qquad \max_{t_n \in \bar{Z}_N} |y(t_n) - \tilde{u}(t_n)| = O(h^\alpha)$$

where $\alpha := \min(m+p, s_0+1, s_1)$ *and y is the solution of linear Volterra integral equation* (3.1).

Proof. We shall prove that formula (3.3) holds, the proof of (3.4) is analogous.

The collocation equation for $\hat{u} \in S^{(d)}_{m+d}(Z_N)$ (equations (2.4)), holds only at the collocation points $X(N)$. It can be written in the following form:

$$(3.5)\qquad \hat{u}(t) = f(t) + \int_0^t K(t,s)\,u(s)\,ds - \hat{\delta}(t),\ t \in I,$$

where δ denotes a suitable function, subsequently called the defect function. This function has the form:

$$(3.6)\qquad \hat{\delta}(t_{n,j}) = h_n E^{(j)}_{n,n}[u_n] + \sum_{i=0}^{n-1} h_i E^{(j)}_{n,i}[u_i], \text{ for all } t_{n,j} \in X(N).$$

Subtraction of (3.5) from (3.1) yields a second-kind integral equation for the error function:

$$(3.7)\qquad \hat{e}(t) = \hat{\delta}(t) + \int_0^t K(t,s)\,\hat{e}(s)\,ds,\ t \in I,$$

whose solution is given by:

$$(3.8)\qquad \hat{e}(t) = \hat{\delta}(t) + \int_0^t R(t,s)\,\hat{\delta}(s)\,ds,\ t \in I,$$

where $R(t,s)$ denotes the resolvent kernel for $K(t,s)$. Substituting $t = t_n$ in (3.8), $n = 1,2,\ldots,N$ it follows:

$$(3.9)\qquad \hat{e}(t_n)=\hat{\delta}(t_n)+\int_0^{t_n} R(t_n,s)\hat{\delta}(s)\,ds = \hat{\delta}(t_n)+\sum_{i=0}^{n-1} h_i \int_0^1 R(t_n,t_i+\tau h_i)\hat{\delta}(t_i+\tau h_i)\,d\tau.$$

If each integral from equation (3.9) is approximated with an interpolator m-point quadrature formula based on the abscissas $\{t_{i,l}\}_{l=\overline{1,m}}$ for all $i = 0,1,\dots,N-1$ we are led to:

$$(3.10)\qquad \hat{e}(t_n)=\hat{\delta}(t_n)+\sum_{i=0}^{n-1} h_i\left(\sum_{l=1}^{m} b_l R(t_n,t_{i,l})\hat{\delta}(t_{i,l})+E_{n,i}\right),$$

where $E_{n,i}$, represents the error of the interpolator quadrature formula used, and b_l, $l = 1,2,\dots,m$ are the weight of these formulas.

By the hypothesis $c_m = 1$ and (3.2) we have $|E_{n,i}| \le C_i\ h^{m+p}$ and by (3.6) we get $|\hat{\delta}(t_{n,j})| \le Qh^{s_0+1}+Ph^{s_1}$ (C_i, Q, P are the positive constants). Using these estimations in equation (3.9) we obtain that

$$|\hat{e}(t_n)| \le Ch^{\alpha},\ \text{with}\ \alpha = \min\{m+p, s_0+1, s_1\}.$$

Remark 3.2. Theorem 3.1 can be proved using Theorem 2.3 stated above and Theorem 3.1 from [5].

By Theorem 3.1 one observes that the local superconvergence on $\overline{Z}_N$ is closely connected with the choice of collocation parameters $\{c_j\}$ (see [1], [2], [3], [4]), the relation between their number of the coefficients of the approximate solution determined from the smooth conditions and with the choice of the quadrature formulas (2.1), respectively (2.30). If the parameter $c_m = 1$, then the number of p cannot exceed m-1, and the convergence order cannot exceed the value $2m$-1. The convergence order $\alpha = 2m$-1 can be obtained when we choose $m \ge d+2$, $\{c_j\}_{j=\overline{1,m}}$ – m Radau II points for (0,1] and the quadrature formulas used are such that s_0+1 and s_1 are greater than $2m$-1 (see [1], [3], [5]).

If in the quadrature formulas (2.1) and (2.30) we consider $\mu_0 = \mu_1 = m$ and $d_j = c_j$, $d_{j,l} = c_jc_l$ for $j,l = 1,\dots,m$ and then we obtain an algorithm for which the local superconvergence order is given in the following theorem.

THEOREM 3.2. *Let the assumption of* Theorem 3.1 *hold. If in quadrature formulas* (2.1) *and* (2.30) *we consider* $\mu_0 = \mu_1 = m$ *and* $d_j = c_j$, $d_{j,l} = c_jc_l$ *for* $j,l =$ *1,...,m, then the following estimations hold:*

$$(3.11)\qquad \max_{t_n\in\overline{Z}_N} |y(t_n)-\hat{u}(t_n)| = O(h^{m+p})$$

$$(3.12)\qquad \max_{t_n\in\overline{Z}_N} |y(t_n)-\tilde{u}(t_n)| = O(h^{m+p})\quad (\text{as } h\searrow 0,\ Nh\le\gamma T),$$

where y is the solution of linear Volterra integral equation (3.1).

Proof. If $\mu_0 = \mu_1 = m$ and $d_j = c_j$, $d_{j,l} = c_jc_l$ for $j,l = 1,\dots,m$, then by (3.2) follows that the convergence order for quadrature formulas (2.1) and (2.30) is $s_0 = s_1 = m+p$, i.e. there exists the positive constants C_1, C_2 such that:

$$(3.13a)\qquad \left|E_{n,i}^{(j)}[u_i]\right| \le C_1h^{m+p},\ i=0,\dots,n\ \left(\left|\widetilde{E}_{n,n}^{(j)}[u_n]\right| \le C_1h^{m+p}\right),$$

and

$$(3.13b)\qquad |E_{n,i}| \le C_2h^{m+p},\ i=0,\dots,n$$

for all $j = 1,\dots,m$ ($n = 0,1,\dots,N-1$).

From (3.6) and (3.13) we get $|\hat{\delta}(t_{n,j})| \le C_3h^{m+p}$, and using this estimation in equation (3.10) we obtain that:

$$|\hat{e}(t_n)| \le Ch^{m+p},\ n=0,1,\dots,N-1.$$

COROLLARY 3.3. *Let the assumption of* Theorem 3.2 *hold, then if the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the zeros of* $P_{m-1}(2s-1) - P_m(2s-1)$ *(i.e the Radau II points for (0,1]), then in* Theorem 3.2 *we have* $p = m-1$, *i.e.:*

$$\max_{t_n\in\overline{Z}_N} |\hat{e}(t_n)| = O(h^{2m-1}),\quad (\text{as } h\searrow 0,\ Nh\le\gamma T)$$

$$\max_{t_n\in\overline{Z}_N} |\tilde{e}(t_n)| = O(h^{2m-1}),\quad (\text{as } h\searrow 0,\ Nh\le\gamma T).$$

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