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ON THE CONVERGENCE OF AN ITERATIVE PROCEEDING OF CHEBYSHEV TYPE

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Let us consider the operational equation:

(1) $$f(x) = \theta$$

where $f: X \to X$; X and Y are normed linear spaces and θ is the null element of the space Y.

Newton-Kantorovich's iterative method for the approximation of the solution of equation (1) consists of a sequence $(x_n)_{n\in \mathbb{N}} \subseteq X$, beginning with an arbitrary element $x_0 \in X$, based on the relation of recurrence:

(2) $$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n), \, n \in \mathbb{N}.$$

It is known that this iterative method has the order 2, which means that in certain conditions imposed to x_0 we have the inequality:

(3) $$\|f(x_n)\| \leq C \|f(x_0)\|^{p^n}$$

with p=2 and C constant.

If in the relation of recurrence (2) we add an adequate term of correction we will obtain Chebyshev's iterative method, method for which the relation of recurrence is:

(4) $$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n) - \frac{1}{2}[f'(x_n)]^{-1} f''(x_n) h_n^2;$$

where

$$h_n = [f'(x_n)]^{-1} f(x_n).$$

Chebyshev's method is faster convergent than Newton's, relation (3) being satisfied for p=3 (but the conditions on the initial element x_0 will be, in this case, stronger).

Obviously, in relations (2) and (4) f' and f'' represent the Fréchet derivatives of order 1 and 2 respectively, of nonlinear mapping f. We will suppose now the existence of these derivatives.

If we denote by $(X,Y)^*$ the set of the linear and continuous mappings defined on X with values in Y, there results that for every $n\in\mathbb{N}$, $[f'(x_n)]^{-1}\in(X,Y)^*$ and thus the application of methods (2) and (4) requires for every $n\in\mathbb{N}$ the inversion of a linear operator, that is the resolution of a linear equation.

This drawback can be eliminated by the introduction of a second sequence $(A_n)_{n\in N}\subseteq(Y,X)^*$ and the approximation by this sequence, simultaneously with the solution $\bar{x}$, of the mapping $[f'(\bar{x})]^{-1}$.

Like in the papers [1], [2], [3], [4], [5] let $p\in\mathbb{N}$ and let mapping $S_{p+1}:(X,Y)^*\times(Y,X)^*\to(Y,X)^*$ be defined for $A\in(X,Y)^*$ and $A_0\in(Y,X)^*$ by:

$$S_{p+1}(A,A_0)=A_0\sum_{k=0}^{p}(I-AA_0)^k,$$

where I is the identical mapping of the space Y. If $A^{-1}\in(Y,X)^*$ exists, $S_{p+1}(A,A_0)$ will be called the $p+1$ approximent of A^{-1} with the aid of A_0.

The sequence $(A_n)_{n\in N}$ defined by $A_{n+1}=S_{p+1}(A,A_n)$ verifies the inequality:

$$\|I-AA_n\|\le\|I-AA_0\|^{(p+1)^n}$$

from where we infer the fact that if $\|I-AA_0\|<1$, there results the existence of the mapping A^{-1} which is obtained as a limit of the sequence $(A_n)_{n\in\mathbb{N}}$ the speed of convergence having the order $p+1$.

Combining method (2) with the simultaneous approximation of the mapping $[f'(\bar{x})]^{-1}$ we obtain the method defined by the following relations:

$$\text{(5)}\qquad\begin{cases}x_{n+1}=x_n-S_{p+1}(f'(x_n),A_n)f(x_n)\\ A_{n+1}=S_{q+1}(f'(x_{n+1}),A_n)\end{cases},$$

$p,q\in\mathbb{N}$, $x_0\in X$ and $A_0\in(Y,X)^*$ being the arbitrary elements. This method was studied in detail in papers [2], [3], [4].

Let us apply the same proceeding to method (4). We will obtain the following variant of the Chebyshev method:

$$\text{(6)}\qquad\begin{cases}D_n=S_{p+1}(f'(x_n),A_n)\\ x_{n+1}=x_n-D_nf(x_n)-\frac{1}{2}D_nf''(x_n)\{D_nf(x_n)\}^2\\ A_{n+1}=S_{q+1}(f'(x_{n+1}),A_n)\end{cases}.$$

Here, too, $p,q\in\mathbb{N}$, $x_0\in X$ and $A_0\in(Y,X)^*$. The convergence of method (6) constitutes the subject of the present paper.

Denoting by $B(x_0,R)$ the ball with the centre in x_0 and having the radius R we have the following:

THEOREM 1. *If $p\ge1$, $q\ge2$, X, and Y are Banach spaces $x_0\in X$, $A_0\in(Y,X)^*$ and $R>0$ and the following conditions are fulfilled:*

i) *f admits the Fréchet derivatives up to the third order, the third order included, the application $f'(x)$ being inversable on every point of the ball $B(x_0,R)$, existing $L,M>0$ so that:*

$$\text{(7)}\qquad\|f(x)\|\le L,\ \|f''(x)\|\le L,\ \|f'''(x)\|\le L\text{ and }\left\|[f'(x)]^{-1}\right\|\le M$$

for every $x\in B(x_0,R)$;

ii) $$d=\max\left\{\frac{1}{C_1}\|f(x_0)\|,\frac{1}{C_2}\|I-f'(x_0)A_0\|\right\}<1,$$

$$R\ge2B(p+1)u\frac{d}{1-d^2},$$

where C_1 and C_2 verify the system:

$$\text{(8)}\qquad\begin{cases}(vC_1^2+uC_2^{p+1})\le1\\ (C_2+wC_1)^{q+1}\le C_2\end{cases},$$

and

$$u=1+2(p+1)^2L^2M^2,$$

$$v=\frac{4}{3}LM(p+1)^3u^3+4L^2M^4(p+1)^4(u+1),$$

$$w=4LM^2(p+1);$$

then:

j) *the sequences $(x_n)_{n\in\mathbb{N}}\subseteq X$ and $(A_n)_{n\in\mathbb{N}}\subseteq(Y,X)^*$ generated by relations (6) are convergent, $\bar{x}=\lim_{n\to\infty}x_n\in B(x_0,R)$ is the solution of equation (1), and $\bar{A}=[f'(\bar{x})]^{-1}=\lim_{n\to\infty}A_n$*

jj) *the following evaluations of the error of approximation holds:*

$$\|x_{n+1}-x_n\|\le2M(p+1)uC_1d^{3^n},$$

$$\|\bar{x}-x_n\|\le2M(p+1)uC_1\frac{d^{3^n}}{1-d^{2\cdot3^n}},$$

$$\|A_{n+1}-A_n\| \le 2M\frac{\alpha d^{3^n}-\left(\alpha d^{3^n}\right)^{q+1}}{1-d^{3^n}},\quad \alpha = C_2 + wC_1,$$

$$\|\bar{A}-A_n\| \le \frac{2M}{1-d^{3^n}}\left[\frac{\alpha d^{3^n}}{1-d^{2\cdot 3^n}} - \frac{\left(\alpha d^{3^n}\right)^{q+1}}{1-d^{2(q+1)3^n}}\right];\quad n\in\mathbb{N}.$$

Proof. We will prove that for every $n \in \mathbb{N}$ the following propositions are true:

a) $x_n \in B(x_0, R)$,

b) $\rho_n = \|f(x_n)\| \le C_1 d^{3^n}$ and $\delta_n = \|I - f'(x_n)A_n\| \le C_2 d^{3^n}$

c) $\|A_n\| \le 2B$.

Evidently, the propositions a) - c) are true for $n = 0$, using for this the hypothesis ii). In case the proposition c), because:

$$\|A_0\| \le \left\|[f'(x_0)]^{-1}\right\|\left(1+\|I-f'(x_0)A_0\|\right) \le M(1+C_2),$$

we infer from system (8) the fact:

$$C_2^{p+1} \le \frac{1}{u}$$

and as $u>1$ we deduce $C_2 < 1$, so $\|A_0\| < 2M$.

Let us suppose that the relations a) - c) are true for every $n \le m$ and we have to prove that they are true for $n=m+1$.

In relations (6) there appears the aiding sequence $(D_n)_{n\in\mathbb{N}} \subseteq (Y,X)^*$.

For every $n \le m$ we have:

$$\|D_n\| = \|S_{p+1}(f'(x_n),A_n)\| = \left\|A_n\sum_{k=0}^{p}(I-f'(x_n)A_n)^k\right\| \le$$

$$\le \|A_n\|\sum_{k=0}^{p}\|I-f'(x_n)A_n\|^k \le \|A_n\|^k\sum_{k=0}^{p}C_2^k d^{k\cdot 3^n}.$$

As $C_2 < 1$, $d < 1$, $\|A_n\| \le 2M$; we infer the fact that: $\|D_n\| \le 2M(p+1)$.

Thus:

$$\|x_{n+1}-x_n\| = \left\|D_n\left\{I+\frac{1}{2}f''(x_n)(D_nf(x_n),D_n)\right\}f(x_n)\right\| \le$$

$$\le \|D_n\|\cdot\|f(x_n)\|\cdot\left\|I+\frac{1}{2}f''(x_n)(D_nf(x_n),D_n)\right\| \le$$

$$\le \|D_n\|\cdot\|f(x_n)\|\left(1+\frac{1}{2}\|f''(x_n)\|\cdot\|D_n\|^2\cdot\|f(x_n)\|\right) \le$$

$$\le 2M(p+1)\left(1+2L^2M^2(p+1)^2\right)\|f(x_n)\| \le$$

$$\le 2M(p+1)\left[1+2L^2M^2(p+1)^2\right]C_1d^{3^n} = 2M(p+1)C_1ud^{3^n};$$

So:

$$\|x_{m+1}-x_0\| \le \sum_{i=0}^{m}\|x_{i+1}-x_i\| \le 2M(p+1)C_1u\sum_{i=0}^{m}d^{3^i}$$

From the fact that $3^k-1=(3-1)(1+3+\ldots+3^{k-1}) > 2k$ and $d<1$ we have:

$$\sum_{i=0}^{m}d^{3^i} = d\sum_{i=0}^{m}d^{3^i-1} < d\sum_{i=0}^{m}\left(d^2\right)^i < \frac{d}{1-d^2};$$

so $\|x_{m+1}-x_0\| \le 2Mu(p+1)\frac{d}{1-d^2} \le R$, so that $x_{m+1} \in B(x_0,R)$.

Then it is obvious that:

$$\|f(x_{m+1})\| \le$$

(10) $$\le \left\|f(x_{m+1})-f(x_m)-f'(x_m)(x_{m+1}-x_m)-\frac{f''(x_m)}{2!}(x_{m+1}-x_m)^2\right\| +$$

$$+\left\|f(x_m)+f'(x_m)(x_{m+1}-x_m)+\frac{f''(x_m)}{2!}(x_{m+1}-x_m)^2\right\|.$$

We denote by A_m and B_m the two terms from the second member of inequality (10). First, using Taylor's formula we have the following:

$$A_m = \left\|f(x_{m+1})-f(x_m)-f'(x_m)(x_{m+1}-x_m)-\frac{f''(x_m)}{2!}(x_{m+1}-x_m)^2\right\| \le$$

(11) $$\le \frac{1}{3!}\sup_{t\in[0,1]}\|f'''(x_m+t(x_{m+1}-x_m))\|\cdot\|x_{m+1}-x_m\|^3 \le$$

$$\le \frac{L}{6}[2M(p+1)u]^3\|f(x_m)\|^3 = \frac{4LM^3(p+1)^3u^3}{3}\|f(x_m)\|^3$$

Then:

$$f'(x_m)(x_{m+1}-x_m) = -f'(x_m)D_m\left[f(x_m)+\frac{1}{2}f''(x_m)(D_mf(x_m))^2\right].$$

So that:

$$f(x_m)+f'(x_m)(x_{m+1}-x_m)=f(x_m)+\frac{1}{2}f''(x_m)(D_mf(x_m))^2-$$

$$-\frac{1}{2}f''(x_m)(D_mf(x_m))^2-f'(x_m)D_m\left[f(x_m)+\frac{1}{2}f''(x_m)(D_mf(x_m))^2\right]=$$

$$=(I-f'(x_m)D_m)\left[f(x_m)+\frac{1}{2}f''(x_m)(D_mf(x_m))\right]-\frac{1}{2}f''(x_m)(D_mf(x_m))^2.$$

We also have:

$$\frac{f''(x_m)}{2!}(x_{m+1}-x_m)^2=\frac{f''(x_m)}{2}\left\{D_mf(x_m)+\frac{1}{2}D_mf''(x_m)(D_mf(x_m))^2\right\}^2=$$

$$=\frac{f''(x_m)}{2}(D_mf(x_m))^2+\frac{f''(x_m)}{2}\left(D_mf(x_m),D_mf''(x_m)(D_mf(x_m))^2\right)+$$

$$+\frac{f''(x_m)}{8}\left(D_mf''(x_m)(D_mf(x_m))^2\right)^2.$$

Thus:

$$f(x_m)+f'(x_m)(x_{m+1}-x_m)+\frac{f''(x_m)}{2!}(x_{m+1}-x_m)^2=$$

$$=(I-f'(x_m)D_m)\left[f(x_m)+\frac{1}{2}f''(x_m)(D_mf(x_m))^2\right]+$$

$$+\frac{f''(x_m)}{2}\left(D_mf(x_m),D_mf''(x_m)(D_mf(x_m))^2\right)+\frac{f''(x_m)}{8}\left(D_mf''(x_m)(D_mf(x_m))^2\right)^2$$

Because:

$$I-f'(x_m)D_m=I-f'(x_m)A_m\sum_{k=0}^{p}(I-f'(x_m)A_m)^k=(I-f'(x_m)A_m)^{p+1}$$

we will have:

$$B_m\le\|I-f'(x_m)A_m\|^{p+1}\|f(x_m)\|\left(1+\frac{\|f''(x_m)\|}{2}\|D_m\|^2\|f(x_m)\|\right)+$$

$$+\frac{\|f''(x_m)\|^2}{2}\|D_m\|^4\|f(x_m)\|^3+\frac{\|f''(x_m)\|^3}{8}\|D_m\|^6\|f(x_m)\|^4\le$$

$$\le\|I-f'(x_m)A_m\|^{p+1}\|f(x_m)\|\left(1+2M^2L^2(p+1)^2\right)+$$

(12) $$+\left[8L^2M^4(p+1)^4+8L^4M^6(p+1)^6\right]\|f(x_m)\|^3$$

From (10), (11) and (12) we obtain:

$$\|f(x_{m+1})\|\le\frac{4}{3}LM^3(p+1)^3u^3\|f(x_m)\|^3+$$

(13) $$+u\|I-f'(x_m)A_m\|^{p+1}\cdot\|f(x_m)\|+8L^2M^4(p+1)^4\left[1+M^2L^2(p+1)^2\right]\cdot$$

$$\cdot\|f(x_m)\|=v\|f(x_m)\|^3+u\|I-f'(x_m)A_m\|^{p+1}\cdot\|f(x_m)\|.$$

We also have:

$$I-f'(x_{m+1})A_{m+1}=I-f'(x_{m+1})A_m\sum_{k=0}^{q}(I-f'(x_{m+1})A_m=$$

$$=(I-f'(x_{m+1})A_m)^{q+1},$$

it is obvious that:

$$\|I-f'(x_{m+1})A_m\|\le\|I-f'(x_m)A_m\|+\|f'(x_{m+1})-f'(x_m)\|\cdot\|A_m\|\le$$

$$\le\|I-f'(x_m)A_m\|+\sup_{t\in[0,1]}\|f''(x_m+t(x_{m+1}-x_m))\|\cdot\|x_{m+1}-x_m\|\cdot\|A_m\|\le$$

(14) $$\le\|I-f'(x_m)A_m\|+L\cdot 2M\cdot 2M(p+1)u\|f(x_m)\|,$$

$$\|I-f'(x_{m+1})A_{m+1}\|\le\left(\|I-f'(x_m)A_m\|+w\|f(x_m)\|\right)^{q+1}$$

Taking into consideration the significances of ρ_m and δ_m inequalities (13) and (14) will be written:

(15) $$\begin{cases}\rho_{m+1}\le v\rho_m^3+u\rho_m\delta_m^{p+1}\\ \delta_{m+1}\le(\delta+w\rho_m)^{q+1}.\end{cases}$$

But $\rho_m\le C_1d^{3^m}$ and $\delta_m\le C_2d^{3^m}$ where C_1 and C_2 are the solutions of system (8). As $p\ge 1$ we obtain:

$$\rho_{m+1}\le vC_1^3d^{3^{m+1}}+uC_1C_2^{p+1}d^{3^m+(p+1)3^m}=$$

$$=C_1\left[vC_1^2+uC_2^{p+1}d^{(p-1)3^m}\right]d^{3^{m+1}}\le C_1d^{3^{m+1}}$$

and

$$\delta_{m+1}\le\left(C_2d^{3^m}+wC_1d^{3^m}\right)^{q+1}=(C_2+wC_1)^{q+1}d^{(q+1)3^m}\le C_2d^{(q+1)3^m}.$$

But $q \geq 2$ so that $(q+1)3^m \geq 3^{m+1}$ and $\delta_{m+1} \leq C_2 d^{3^{m+1}}$.

So the propositions b) are true for $n = m+1$.

The propositon c) results identically with the case $n=0$.

So, based on the principle of mathematical induction the relations a) - c) are true for every $n \in \mathbb{N}$.

We will prove now that the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(16)
$$\|x_{n+m} - x_n\| \leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \leq 2M(p+1)C_1 u \sum_{i=n}^{n+m-1} d^{3^i} = \\ = 2M(p+1)C_1 u d^{3^n} \sum_{j=0}^{m-1} d^{3^{n+j}-3^n} < 2M(p+1)C_1 u \frac{d^{3^n}}{1-d^{2\cdot 3^n}}.$$

But $d<1$ so that $\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$, hence the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent being in the Banach space X.

If $\bar{x} = \lim_{n\to\infty} x_n$, then from (16) we will obtain:

$$\|\bar{x} - x_n\| \leq 2M(p+1)uC_1 \frac{d^{3^n}}{1-d^{2\cdot 3^n}},$$

from where if $n=0$ we deduce $\|\bar{x} - x_0\| \leq R$, so $\bar{x} \in B(x_0, R)$.

Refering to the sequence $(A_n)_{n\in\mathbb{N}} \subseteq (Y,X)^*$ we have:

$$\|A_{n+1} - A_n\| \leq \|A_n\| \sum_{k=1}^{q} \|I - f'(x_{n+1})A_n\|^k \leq \\ \leq 2B \sum_{k=1}^{q} (\delta_m + w\rho_m)^k \leq 2B \sum_{k=1}^{q} \left[(C_2 + wC_1)d^{3^n}\right]^k$$

As $\alpha = C_2 + wC_1$ we deduce that:

$$\|A_{n+1} - A_n\| \leq 2B \frac{\alpha d^{3^n} - \left(\alpha d^{3^n}\right)^{q+1}}{1-\alpha d^{3^n}} \text{ and so:}$$

(17)
$$\|A_{n+m} - A_n\| \leq \sum_{i=n}^{n+m-1} \|A_{i+1} - A_i\| \leq 2B \sum_{i=n}^{n+m-1} \frac{\alpha d^{3^i} - \left(\alpha d^{3^i}\right)^{q+1}}{1-\alpha d^{3^i}} \leq \\ \leq \frac{2B}{1-\alpha d^{3^n}} \left[\alpha \sum_{i=n}^{n+m-1} d^{3^i} - \alpha^{q+1} \sum_{i=n}^{n+m-1} \left(d^{(q+1)3^i}\right)\right] < \\ < \frac{2B}{1-\alpha d^{3^n}} \left[\frac{\alpha d^{3^n}}{1-\alpha d^{2\cdot 3^n}} - \frac{\left(\alpha d^{3^n}\right)^{q+1}}{1-d^{2(q+1)3^n}}\right].$$

From (17) we deduce that $\lim_{n\to\infty} \|A_{n+m} - A_n\| = 0$, so the sequence $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(Y,X)^*$ so it is convergent in this space.

If $\bar{A} = \lim_{n\to\infty} A_n$, then from (17) we deduce:

$$\|\bar{A} - A_n\| \leq \frac{2B}{1-\alpha d^{3^n}} \left[\frac{\alpha d^{3^n}}{1-\alpha d^{2\cdot 3^n}} - \frac{\left(\alpha d^{3^n}\right)^{q+1}}{1-d^{2(q+1)3^n}}\right].$$

So the theorem is proved.

To simplify the method it is good to consider for $p, q \in \mathbb{N}$ the smallest possible values, which are $p=1$ and $q=2$. Then in the method the following will appear:

$$S_2(f'(x_n), A_n) = A_n(2I - f'(x_n)A_n) \text{ and}$$
$$S_3(f'(x_{n+1}), A_n) = A_n\left[3I - 3f'(x_{n+1})A_n + (f'(x_{n+1})A_n)^2\right].$$

In this case the iterative proceeding (6) becomes:

(18)
$$\begin{cases} D_n = A_n(2I - f'(x_n)A_n), \\ x_{n+1} = x_n - D_n f(x_n) - \frac{1}{2} D_n f''(x_n)(D_n f(x_n))^2, \\ A_{n+1} = A_n\left[3I - 3f'(x_{n+1})A_n + (f'(x_{n+1})A_n)^2\right]; \end{cases} \quad n \in \mathbb{N}$$

The constants u, v, w will become:

(19)
$$\begin{cases} u = 1 + 8L^2M^2, \\ v = \frac{32LM^3}{3}\left(1+8L^2M^2\right)^3 + 128L^2M^4\left(1+4L^2M^2\right), \\ w = 8LM^2\left(1+8L^2M^2\right), \end{cases}$$

and the radius of the ball on which the conditions over the application f are imposed is given by the inequality:

$$R \geq 4Mu \frac{d}{1-d^2}.$$

The constant C_1 and C_2 will be solutions of the system:

$$(20)\qquad \begin{cases} vC_1^2 + uC_2^2 \le 1 \\ (C_2 + wC_1)^3 \le C_2 \end{cases}$$

Let us examine the determination of a convenient solution of system (20). In the first inequality from (20) we suppose to be satisfied the equality and we will deduce:

$$C_1 = \sqrt{\frac{1-uC_2^2}{v}}$$

so for C_2 we will obtain:

$$(21)\qquad \left(C_2 + w\sqrt{\frac{1-uC_2^2}{v}}\right)^3 \le C_2.$$

Let us put in (21) $C_2 = x^3 \in]0,1[$ and we will obtain:

$$x_3 + w\sqrt{\frac{1-ux^6}{v}} \le x,$$

inequality which is equivalent to:

$$(v+uw^6)x^6 - 2vx^4 + vx^2 - w^2 \ge 0.$$

If now we put here $x^2 = y$ we obtain:

$$(22)\qquad \varphi(y) = (v+uw^2)y^3 - 2vy^2 + vy - w^2 \ge 0$$

If we calculate the derivatives of the order 1 and 2 of the function φ we obtain:

$$\varphi'(y) = 3(v+uw^2)y^2 - 4vy + v \text{ and}$$

$$\varphi''(y) = 6(v+uw^2)y^2 - 4v.$$

It is obvious that:

$$\varphi(0) = -w^2 < 0,\quad \varphi(1) = (u-1)w^2 > 0,$$

$$\varphi'(1) = uw^2 > 0,\quad \varphi''(1) = 2(v+3uw^2) > 0.$$

So the equation $\varphi(y) = 0$ has at least one solution in the interval $[0,1]$ and the first Newton approximation relative at the function φ and the point $y=1$ is superior to the largest of these solutions. So, this approximation will represent a convenient solution of the inequality (22) and this will be:

$$y_1 = 1 - \frac{\varphi(1)}{\varphi'(1)} = 1 - \frac{(u-1)w^2}{uw^2} = \frac{1}{u}.$$

As $C_2 = y^{\frac{3}{2}}$ it results that $C_2 = \frac{1}{u\sqrt{u}}$ and accordingly $C_1 = \frac{1}{u}\sqrt{\frac{u^2-1}{v}}$

So, we have the following:

COROLLARY 2. *If X and Y are Banach spaces, $x_0 \in X$, $A_0 \in (Y,X)^*$, $R > 0$ and the following conditions are fulfilled:*

i) f admits Fréchet derivatives up to the third order, the third order included, the mapping $f'(x)$ being inversable on every point of the ball $B(x_0, R)$, existing L, M so that inequalities (7) are fulfilled for every $x \in B(x_0, R)$;

ii) u, v, w being the real numbers given by relation (19); x_0, A_0 are solutions of the inequalities:

$$\|f(x_0)\| < \frac{1}{u}\sqrt{\frac{u^2-1}{v}},$$

$$\|I - f'(x_0)A_0\| < \frac{1}{u\sqrt{u}},$$

and

$$R \ge 4Mu\frac{d}{1-d^2},$$

where:

$$d = \max\left\{\frac{u\sqrt{v}\|f(x_0)\|}{u^2-1}, u\sqrt{u}\|I - f'(x_0)A_0\|\right\} < 1$$

then the conclusion j) of theorem 1 holds and we have the following estimates:

$$\|x_{n+1} - x_n\| \le 4M\sqrt{\frac{u^2-1}{v}}\cdot d^{3^n},$$

$$\|\bar{x} - x_n\| \le 4M\sqrt{\frac{u^2-1}{v}}\frac{d^{3^n}}{1-d^{2\cdot 3^n}},$$

$$\|A_{n+1} - A_n\| \le 2M\frac{\alpha d^{3^n} - \alpha^3 d^{3^{n+1}}}{1-d^{3^n}},$$

$$\|\bar{A} - A_n\| \le \frac{2M}{1-d^{3^n}}\left[\frac{\alpha d^{3^n}}{1-d^{2\cdot 3^n}} - \frac{\alpha^3 d^{3^{n+1}}}{1-d^{2\cdot 3^{n+1}}}\right],$$

where: $\alpha = \frac{1}{u\sqrt{u}} + \frac{w}{u}\sqrt{\frac{u^2-1}{v}}.$

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