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# APPLICATION OF DIVIDED DIFFERENCES TO THE STUDY OF MONOTONICITY OF A SEQUENCE OF D. D. STANCU POLYNOMIALS

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## 1. INTRODUCTION

In the paper [6] D. D. Stancu has used a probabilistic method to construct a linear positive polynomial operator $L_{m,r}^{\alpha,\beta}$ of Bernstein type, depending on a non-negative integer parameter $r$ ($2r < m$) and on two real parameters $\alpha$ and $\beta$, such that $0 \leq \alpha \leq \beta$. This operator is defined by means of the following formula:

$$(1)\qquad \left(L_{m,r}^{\alpha,\beta} f\right)(x) = \sum_{k=0}^{m-r} p_{m-r,k}(x)\left[(1-x)f\left(\frac{k+\alpha}{m+\beta}\right) + xf\left(\frac{k+r+\alpha}{m+\beta}\right)\right],$$

where $p_{m-r,k}$ are the fundamental Bernstein polynomials:

$$(2)\qquad p_{m-r,k}(x) = \binom{m-r}{k} x^k (1-x)^{m-r-k}.$$

Special attention was granted to the case of the operator $L_{m,r}^{0,0} = L_{m,r}$. The author proved that the remainder of the approximation formula of a function $f \in C[0,1]$ by $L_{m,r} f$ can be represented by means of divided differences, or in an integral form obtained by using a classical theorem of Peano. Also, the operator $L_{m,r}$ enjoys the variation diminishing property, in the sense of I. J. Schoenberg [3]. In the same paper the orders of approximation in terms of the modulus of continuity of the function $f$ or of its derivative were evaluated and the point spectrum of the operator $L_{m,r}$ was determined. Finally, a quadrature formula which can be constructed by means of this operator was presented.

In this paper we shall use the divided differences as fundamental mathematical tools in the investigation of the monotonicity properties of the sequence $(L_{m,r}f)$, $m = 1, 2, \ldots$

## 2. PRELIMINARY RESULTS

In 1969 D. D. Stancu [5] considered and studied the following generalization of the Bernstein polynomial:

$$(3)\qquad \left(S_n^{a,b} f\right)(x)=\sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+a}{n+b}\right)$$

where $p_{n,k}$ are given at (2) and $0 \le a \le b$.

In our paper [1] we investigated the monotonicity properties of this sequence. At the same time we established a useful formula for the difference of two consecutive terms of the Stancu--Bernstein polynomials. This difference can be expressed under the form:

$$(4)\qquad \left(S_{n+1}^{a,b} f\right)(x)-\left(S_n^{a,b} f\right)(x)=$$

$$=-\frac{nx(1-x)}{(n+b)(n+b+1)}\sum_{k=0}^{n-1}\binom{n-1}{k}\left(\frac{1}{n+b}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]+\right.$$

$$\left.+\frac{a}{k+1}\left[\frac{k+1+a}{n+1+b},\frac{k+1+a}{n+b};f\right]-\frac{b-a}{n-k}\left[\frac{k+a}{n+b},\frac{k+1+a}{n+1+b};f\right]\right)x^k(1-x)^{n-k-1}+\left(U_n^{a,b} f\right)(x)$$

where the brackets represent the symbol for divided differences and

$$(5)\qquad \left(U_n^{a,b} f\right)(x)=\left(f\left(\frac{a}{n+1+b}\right)-f\left(\frac{a}{n+b}\right)\right)(1-x)^{n+1}+\left(f\left(\frac{n+1+a}{n+1+b}\right)-f\left(\frac{n+a}{n+b}\right)\right)x^{n+1}.$$

It is easy to verify that by starting from (3) we can obtain the following representation for $L_{m,r}^{\alpha,\beta}$ defined at (1):

$$\left(L_{m,r}^{\alpha,\beta} f\right)(x) = (1-x)\left(S_{m-r}^{\alpha,\beta+r} f\right)(x) + x\left(S_{m-r}^{\alpha+r,\beta+r} f\right)(x).$$

In the above if we set $\alpha=\beta=0$ we obtain:

$$(6)\qquad \left(L_{m,r} f\right)(x) = (1-x)\left(S_{m-r}^{0,r} f\right)(x) + x\left(S_{m-r}^{r,r} f\right)(x).$$

Before establishing the main result we present some identities which will be used later.



LEMMA 1. *If we define the sums:*

$$(7)\qquad B_f(m-r,x) = \sum_{k=0}^{m-r-1}\frac{r}{m-r-k}\binom{m-r-1}{k}\left[\frac{k}{m},\frac{k+1}{m+1};f\right]x^k(1-x)^{m-r-k-1}$$

*and*

$$(8)\qquad C_f(m-r,x)=\sum_{k=0}^{m-r-1}\frac{r}{k+1}\binom{m-r-1}{k}\left[\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]x^k(1-x)^{m-r-k-1},$$

*then the following identities hold:*

$$(1-x)B_f(m-r,x)=\frac{r}{m-r}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k}{m},\frac{k+1}{m+1};f\right]-\frac{m(m+1)}{m-r}\left(f\left(\frac{m-r+1}{m+1}\right)-f\left(\frac{m-r}{m}\right)\right)x^{m-r}$$

$$xC_f(m-r,x)=\frac{r}{m-r}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k+r}{m+1},\frac{k+r}{m};f\right]-\frac{m(m+1)}{m-r}\left(f\left(\frac{r}{m}\right)-f\left(\frac{r}{m+1}\right)\right)(1-x)^{m-r}.$$

*Proof.* We can write:

$$(1-x)B_f(m-r,x)=r\sum_{k=0}^{m-r-1}\binom{m-r-1}{k}\frac{1}{m-r-k}x^k(1-x)^{m-r-k}\left[\frac{k}{m},\frac{k+1}{m+1};f\right]=$$

$$=\frac{r}{m-r}\sum_{k=0}^{m-r-1}\binom{m-r}{k}x^k(1-x)^{m-r-k}\left[\frac{k}{m},\frac{k+1}{m+1};f\right]=$$

$$=\frac{r}{m-r}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k}{m},\frac{k+1}{m+1};f\right]-\frac{r}{m-r}\left[\frac{m-r}{m},\frac{m-r+1}{m+1};f\right]x^{m-r}=$$

$$=\frac{r}{m-r}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k}{m},\frac{k+1}{m+1};f\right]-\frac{m(m+1)}{m-r}\left(f\left(\frac{m-r+1}{m+1}\right)-f\left(\frac{m-r}{m}\right)\right)x^{m-r},$$

since

$$\binom{m-r-1}{k}\frac{1}{m-r-k}=\frac{1}{m-r}\binom{m-r}{k}.$$

In a similar way we find that

$$xC_f(m-r,x)=r\sum_{k=0}^{m-r-1}\binom{m-r-1}{k}\frac{1}{k+1}\left[\frac{k+r+1}{m},\frac{k+r+1}{m+1};f\right]x^{k+1}(1-x)^{m-r-k-1}=$$

$$=\frac{r}{m-r}\sum_{k=0}^{m-r-1}\binom{m-r}{k+1}x^{k+1}(1-x)^{m-r-k-1}\left[\frac{k+r+1}{m},\frac{k+r+1}{m+1};f\right],$$

since

$$\binom{m-r-1}{k}\frac{1}{k+1}=\frac{1}{m-r}\binom{m-r}{k+1}.$$

In the last sum of the equality we set $i=k+1$ and then denoting again the summation index by $k$, we obtain:

$$xC_f(m-r,x)=\frac{r}{m-r}\sum_{k=1}^{m-r}\binom{m-r}{k}x^k(1-x)^{m-r-k}\left[\frac{k+r}{m},\frac{k+r}{m+1};f\right]=$$

$$=\frac{r}{m-r}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k+r}{m},\frac{k+r}{m+1};f\right]-\frac{r}{m-r}\left[\frac{r}{m},\frac{r}{m+1};f\right](1-x)^{m-r}=$$

$$=\frac{r}{m-r}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k+r}{m},\frac{k+r}{m+1};f\right]-\frac{m(m+1)}{m-r}\left(f\left(\frac{r}{m}\right)-f\left(\frac{r}{m+1}\right)\right)(1-x)^{m-r}.$$

This completes the proof of Lemma 1.

LEMMA 2. *The next identity*

$$\left[\frac{k+r}{m},\frac{k+r}{m+1};f\right]-\left[\frac{k}{m},\frac{k+1}{m+1};f\right]=\frac{r}{m}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+r}{m};f\right]+\frac{r-1}{m+1}\left[\frac{k+1}{m+1},\frac{k+r}{m},\frac{k+r}{m+1};f\right]$$

*holds.*

*Proof.* It is obvious that the second side of the identity can be rewritten:

$$\frac{r}{m}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+r}{m};f\right]+\frac{r-1}{m+1}\left[\frac{k+1}{m+1},\frac{k+r}{m},\frac{k+r}{m+1};f\right]=\left[\frac{k+1}{m+1},\frac{k+r}{m};f\right]-\left[\frac{k}{m},\frac{k+1}{m+1};f\right]+$$

$$+\left[\frac{k+r}{m},\frac{k+r}{m+1};f\right]-\left[\frac{k+1}{m+1},\frac{k+r}{m};f\right]=\left[\frac{k+r}{m},\frac{k+r}{m+1};f\right]-\left[\frac{k}{m},\frac{k+1}{m+1};f\right].$$

## 3. MAIN RESULT

In order to study the monotonicity of the sequence $(L_{m,r}f)$ we shall determine the diference of two consecutive terms of the Stancu polynomials.

In addition to (6) and (4) we have:

$$(9)\qquad \left(L_{m+1,r}f\right)(x)-\left(L_{m,r}f\right)(x)=$$

$$=(1-x)\left(\left(S^{0,r}_{m-r+1}f\right)(x)-\left(S^{0,r}_{m-r}f\right)(x)\right)+x\left(\left(S^{r,r}_{m-r+1}f\right)(x)-\left(S^{r,r}_{m-r}f\right)(x)\right)=$$



$$=-\frac{(m-r)x(1-x)^2}{m(m+1)}\sum_{k=0}^{m-r-1}\binom{m-r-1}{k}\left(\frac{1}{m}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]-\right.$$

$$\left.-\frac{r}{m-r-k}\left[\frac{k}{m},\frac{k+1}{m+1};f\right]\right)x^k(1-x)^{m-r-k-1}+(1-x)\left(U^{0,r}_{m-r}f\right)(x)-$$

$$-\frac{(m-r)x^2(1-x)}{m(m+1)}\sum_{k=0}^{m-r-1}\binom{m-r-1}{k}\left(\frac{1}{m}\left[\frac{k+r}{m},\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]+\right.$$

$$\left.+\frac{r}{k+1}\left[\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]\right)x^k(1-x)^{m-r-k-1}+x\left(U^{r,r}_{m-r}f\right)(x)=$$

$$=-\frac{(m-r)x(1-x)}{m^2(m+1)}\sum_{k=0}^{m-r-1}p_{m-r-1,k}(x)\left\{(1-x)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]+\right.$$

$$\left.+x\left[\frac{k+r}{m},\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]\right\}+\frac{(m-r)x(1-x)^2}{m(m+1)}B_f(m-r,x)-$$

$$-\frac{(m-r)x^2(1-x)}{m(m+1)}C_f(m-r,x)+(1-x)\left(U^{0,r}_{m-r}f\right)(x)+x\left(U^{r,r}_{m-r}f\right)(x).$$

In the last equality we used the notations defined at (7) and (8). According to Lemma 1 we can write successively:

$$\left(L_{m+1,r}f\right)(x)-\left(L_{m,r}f\right)(x)=$$

$$=-\frac{(m-r)x(1-x)}{m^2(m+1)}\sum_{k=0}^{m-r-1}p_{m-r-1,k}(x)\left\{(1-x)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]+\right.$$

$$\left.+x\left[\frac{k+r}{m},\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]\right\}+\frac{rx(1-x)}{m(m+1)}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k}{m},\frac{k+1}{m+1};f\right]-$$

$$-\left(f\left(\frac{m-r+1}{m+1}\right)-f\left(\frac{m-r}{m}\right)\right)x^{m-r+1}(1-x)-\frac{rx(1-x)}{m(m+1)}\sum_{k=0}^{m-r}p_{m-r,k}(x)\left[\frac{k+r}{m+1},\frac{k+r}{m};f\right]+$$

$$+\left(f\left(\frac{r}{m}\right)-f\left(\frac{r}{m+1}\right)\right)x(1-x)^{m-r+1}+(1-x)\left(U^{0,r}_{m-r}f\right)(x)+x\left(U^{r,r}_{m-r}f\right)(x).$$

Referring to (5) and taking into account that:

$$(1-x)\left(U^{0,r}_{m-r}f\right)(x)=\left(f\left(\frac{m-r+1}{m+1}\right)-f\left(\frac{m-r}{m}\right)\right)x^{m-r+1}(1-x)$$

and

$$x\left(U^{r,r}_{m-r}f\right)(x) = -\left(f\left(\frac{r}{m}\right) - f\left(\frac{r}{m+1}\right)\right)x(1-x)^{m-r+1},$$

the expression of difference (9) becomes:

$$-\frac{(m-r)x(1-x)}{m^2(m+1)}\sum_{k=0}^{m-r-1} p_{m-r-1,k}(x)\left\{(1-x)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]+\right.$$

$$\left.+x\left[\frac{k+r}{m},\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]\right\}-\frac{rx(1-x)}{m(m+1)}\sum_{k=0}^{m-r} p_{m-r,k}(x)\left\{\left[\frac{k+r}{m+1},\frac{k+r}{m};f\right]-\left[\frac{k}{m},\frac{k+1}{m+1};f\right]\right\}$$

By using Lemma 2 we obtain the following result:

THEOREM 1. *The difference between the Stancu polynomials* $(L_{m+1,r}f)$ *and* $(L_{m,r}f)$ *can be expressed in the form:*

(10)
$$\left(L_{m+1,r}f\right)(x)-\left(L_{m,r}f\right)(x)=$$

$$=\frac{x(x-1)}{m(m+1)}\left\{(m-r)\sum_{k=0}^{m-r-1}p_{m-r-1,k}(x)\left(\frac{1-x}{m}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+r}{m};f\right]+\frac{x}{m}\left[\frac{k+r}{m},\frac{k+r+1}{m+1},\frac{k+r+1}{m};f\right]\right)+\right.$$

$$\left.+r\sum_{k=0}^{m-r}p_{m-r,k}(x)\left(\frac{r}{m}\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+r}{m};f\right]+\frac{r-1}{m+1}\left[\frac{k+1}{m+1},\frac{k+r}{m},\frac{k+r}{m+1};f\right]\right)\right\}.$$

If we choose $r=0$ in Theorem 1 we obtain:

(11)
$$\left(L_{m+1,0}f\right)(x)-\left(L_{m,0}f\right)(x)=\frac{x(x-1)}{m(m+1)}\sum_{k=0}^{m-1}p_{m-1,k}(x)\left[\frac{k}{m},\frac{k+1}{m},\frac{k+1}{m+1};f\right].$$

By choosing $r=1$ in Theorem 1 we obtain the same result:

(12)
$$\left(L_{m+1,1}f\right)(x)-\left(L_{m,1}f\right)(x)=\frac{x(x-1)}{m(m+1)}\sum_{k=0}^{m-1}p_{m-1,k}(x)\left[\frac{k}{m},\frac{k+1}{m},\frac{k+1}{m+1};f\right].$$

Indeed, in this case the expression between braces in relation (10) will be written:

(13)
$$(m-1)\sum_{k=0}^{m-2}p_{m-2,k}(x)\frac{1-x}{m}\left(\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]+\frac{x}{m}\left[\frac{k+1}{m},\frac{k+2}{m+1},\frac{k+2}{m};f\right]\right)+$$

$$+\frac{1}{m}\sum_{k=0}^{m-1}p_{m-1,k}(x)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right].$$



As:

$$\sum_{k=0}^{m-2}xp_{m-2,k}(x)\left[\frac{k+1}{m},\frac{k+2}{m+1},\frac{k+2}{m};f\right]=\sum_{k=1}^{m-1}xp_{m-2,k-1}(x)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]$$

relation (13) becomes:

$$\frac{m-1}{m}(1-x)p_{m-2,0}(x)\left[0,\frac{1}{m+1},\frac{1}{m};f\right]+\sum_{k=1}^{m-2}\left(\frac{m-1}{m}(1-x)p_{m-2,k}(x)+\right.$$

$$\left.+\frac{m-1}{m}xp_{m-2,k-1}(x)+\frac{1}{m}p_{m-1,k}(x)\right)\left[\frac{k}{m},\frac{k+1}{m+1},\frac{k+1}{m};f\right]+$$

$$+\frac{m-1}{m}xp_{m-2,m-2}(x)\left[\frac{m-1}{m},\frac{m}{m+1},1;f\right]+\frac{1}{m}p_{m-1,0}(x)\left[0,\frac{1}{m+1},\frac{1}{m};f\right]+$$

$$+\frac{1}{m}p_{m-1,m-1}(x)\left[\frac{m-1}{m},\frac{m}{m+1},1;f\right].$$

Taking into account the following identities:

$$(1-x)p_{m-2,0}(x)=p_{m-1,0}(x)=(1-x)^{m-1},$$

$$xp_{m-2,m-2}(x)=p_{m-1,m-1}(x)=x^{m-1}$$

$$(m-1)(1-x)p_{m-2,k}(x)+(m-1)xp_{m-2,k-1}(x)+p_{m-1,k}(x)=mp_{m-1,k}(x)$$

they lead us to the desired result.

In these special cases $(L_{m,0}f)$, $(L_{m,1}f)$ reduce to the classical Bernstein polynomials $(B_m f)$ and formula (11) has been established first by D. D. Stancu in paper [4].

We will use the following

DEFINITION (see [2]). *A real-valued function defined on an interval I is called convex of order n on I if all its divided differences of order n+1, on n+2 distinct points of I, are positive. The function is said to be concave of order n on the interval I if all its divided differences of order n+1, on any n+2 distinct points of I, are negative.*

Since on the interval $(0,1)$ we have $p_{n,k}(x)>0$ $(k=0,1,\ldots,n)$ where $n=m-r-1$ respectively $n=m-r$ and $m>2r\geq 0$ from this definition and Theorem 1 there follows:

THEOREM 2. *If the function f is convex (respectively concave) of first order on the interval* $[0,1]$, *then the sequence* $(L_{m,r}f)$ *is decreasing (respectively increasing) on* $(0,1)$, *with respect to the natural number m, such that* $m>2r$.

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