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A CHARACTERIZATION OF REFLEXIVITY

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Let $(X, \|\cdot\|)$ be a real normed space and consider the norm derivatives:

$$(x, y)_{i(s)} := \lim_{t \to 0-(+)} \left(\|y + tx\|^2 - \|y\|^2\right) / 2t.$$

Note that these mappings are well defined on $X \times X$ and the following properties are valid (see also [1] or [2]):

(i) $(x, y)_i = -(-x, y)_s$ if x, y are in X;

(ii) $(x, x)_p = \|x\|^2$ for all x in X;

(iii) $(\alpha x, \beta y)_p = \alpha\beta (x, y)_p$ for all x, y in X and $\alpha\beta \geq 0$;

(iv) $(\alpha x + y, x)_p = \alpha \|x\|^2 + (y, x)_p$ for all x, y in X and $\alpha \in \mathbb{R}$;

(v) $(x + y, z)_p \leq \|x\| \|z\| + (y, z)_p$ for all x, y, z in X;

(vi) the element x in X is Birkhoff orthogonal over y in X (we denote $x \perp y$), i.,e., $\|x + ty\| \geq \|x\|$ for all t in $\mathbb{R}$ iff $(y, x)_i \leq 0 \leq (y, x)_s$;

(vii) the space X is smooth iff $(y, x)_i = (y, x)_s$ for all x, y, in X or iff $(\ ,\ )_p$ is linear in the first variable;

where $p = s$ or $p = i$.

We will use the following well-known result due to R. C. James [3].

THEOREM. *The Banach space X is reflexive iff for any closed hyperplane H in X containing the null vector, there exists an element $u \in X \setminus \{0\}$ so that $u \perp H$.*

The following characterization of reflexivity in terms of convex functions also holds:

THEOREM 1. *Let X be a real Banach space. The following statements are equivalent:*

(i) *X is reflexive;*

(ii) *For every $F: X \to \mathbb{R}$ a convex and continuous mapping on X and for any $x_0 \in X$, there exists an element $u_{F,x_0} \in X$ so that the estimation*

$$F(x) \geq F(x_0) + \left(x - x_0, u_{F,x_0}\right)_i \quad \text{for all } x \in X \tag{1}$$

holds.

Proof. "(i) $\Rightarrow$ (ii)". Let F be a convex and continuous mapping on X. Then F is subdifferentiable on X, i.e., for every $x_0 \in X$ there exists a continuous linear functional f_{x_0} so that

$$F(x) - F(x_0) \geq f_{x_0}(x - x_0) \quad \text{for all } x \text{ in } X. \tag{2}$$

Since X is assumed to be reflexive, hence, by James' theorem, there is an element $w_{F,x_0} \in X \setminus \{0\}$ so that $w_{F,x_0} \perp \mathrm{Ker}(f_{x_0})$.

Because a simple calculation shows that

$$f_{x_0}(x) w_{F,x_0} - f\left(w_{F,x_0}\right) x \in \mathrm{Ker}\left(f_{x_0}\right)$$

for all x in X, hence, by the property (vi), we get that

$$0 \leq \left(f_{x_0}(x) w_{F,x_0} - f_{x_0}\left(w_{F,x_0}\right) x, w_{F,x_0}\right)_s \quad \text{for all } x \in X,$$

which is equivalent, by the above properties of the norm derivatives $(\ ,\ )_p$, with

$$\left(x, u_{F,x_0}\right)_i \leq f_{x_0}(x) \quad \text{for all } x \text{ in } X,$$

where

$$u_{F,x_0} := f_{x_0}\left(w_{F,x_0}\right) w_{F,x_0} / \left\|w_{F,x_0}\right\|^2.$$

Now, by (2) we obtain the estimation (1).

"(ii) $\Rightarrow$ (i)". Let H be a closed hyperplane in X containing the null vector and $f \in X^* \setminus \{0\}$ with Ker $(f) = H$. Then, by (ii), for $F = f$ and $x_0 = 0$, we can find an element u_f in X so that

$$f(x) \geq \left(x, u_f\right)_i \quad \text{for all } x \text{ in } X.$$

Substituting x by $(-x)$ we also have

$$f(x) \leq \left(x, u_f\right)_s \quad \text{for all } x \text{ in } X.$$

Now, we observe that $u_f \neq 0$ (because $f \neq 0$) and then

$$\left(x, u_f\right)_i \leq 0 \leq \left(x, u_f\right)_s \quad \text{for all } x \text{ in } H,$$

i.e., $u_f \perp H$ and by James' theorem we deduce that X is reflexive.

The folowing consequences are interesting too.

COROLLARY1.1. *Let X be a real Banach space. Then X is reflexive iff for every $p: X \to \mathbb{R}$ a continuous sublinear functional on X there exists an element u_p in X so that*

$$p(x) \geq \left(x, u_f\right)_i \text{ for all } x \text{ in } X.$$

COROLLARY. 1.2. (see [2]) *Let X be a real Banach space. Then X is reflexive iff for every $f \in X^*$ there is an element u_f in X so that*

$$\left(x, u_f\right)_i \leq f(x) \leq \left(x, u_f\right) \text{ for all } x \text{ in } X.$$

COROLLARY. 1.3. (see [2]) *Let X be a real Banach space. Then X is smooth and reflexive iff for all $f \in X^*$ there exists an element $u_f \in X$ so that*

$$f(x) = \left(x, u_f\right)_p \text{ for all } x \text{ in } X$$

where $p = s$ or $p = i$.

For other details in connection with the above results, see the recent papers [1] and [2] where further references are given.

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