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A CHARACTERIZATION OF SOME SETS OF THE REAL LINE BY THE FIXED POINT PROPERTY

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In this paper we will characterize the simplest compact sets of the real line, the bounded and closed intervals, using the fixed point property of the continuous functions class. We will introduce the notion of the nearly compact set and we will show that these sets are characterized by the fixed point property of the monotone increasing functions class. In particular we will obtain the fixed point property of the compact sets on the real line.

In this paper we suppose that $M \neq \varnothing$, $M \subseteq R$. We recall the next:

DEFINITION 1. *The set M has the weak fixed point property if every continuous function f: $M \to M$ has a fixed point, i.e. there exists $x \in M$ such that $f(x) = x$.*

THEOREM 1. *The set M has the form $[a, b]$, where a, $b \in \mathbf{R}$, $a \leq b$ if and only if M has the weak fixed point property.*

Proof. If we suppose that $M = [a, b]$, where $a, b \in \mathbf{R}$, $a \leq b$ and f: $[a, b] \to [a, b]$ is an arbitrary continuous function then the function g: $[a, b] \to \mathbf{R}$, $g(x) = f(x) - x$ is also continuous. Because $g(a) = f(a) - a \geq 0$, $g(b) = f(b) - b \leq 0$ and g has the Darboux property, it takes the value $0 \in [g(b), g(a)]$, so there exists $x \in [a; b]$ such that $g(x) = 0$, i.e. $f(x) = x$.

Conversely, let us suppose that M has the weak fixed point property. First we show that M is a connected set. If this is not true then there exists $x_0 \in \mathbf{R}$, $x_0 \notin M$ such that $(-\infty, x_0) \cap M \neq \varnothing$ and $(x_0, +\infty) \cap M \neq \varnothing$. Let us choose the values $x_1 \in (-\infty, x_0) \cap M$ and $x_2 \in (x_0, +\infty) \cap M$ and define the function $f: M \to M$, $f(x) = x_2$ if $x \in (-\infty, x_0) \cap M$ and $f(x) = x_1$, if $x \in (x_0, +\infty) \cap M$. But $x_1 < x_0 < x_2$, so the function f is continuous and has not the weak fixed point property. This contradiction shows that M is a connected set, i. e. M is an interval.

If M is an unbounded interval on right or on left then the following formulas f:$M \to M$, $f(x) = x+1$ or $f(x) = x-1$ defines continuous functions which have not fixed points. It follows that M is a bounded interval.

Now, if we suppose that M has the form (a, b) or $(a, b]$, where a, $b \in \mathbf{R}$, $a<b$ then for an arbitrary fixed number $\alpha \in (0,1)$ the formulas f_α:$(a, b) \to (a, b)$ and

$f'_\alpha:(a, b]\to(a, b]$, $f_\alpha(x)=f'_\alpha(x)=\alpha x+(1-\alpha)a$ defines continuous functions which have not fixed points. Similarly, if M has the form $[a, b)$ then the function $f''_\alpha:[a, b)\to[a, b)$, $a<b$, $f''_\alpha(x)=\alpha.x+(1-\alpha).b$ is continuous but hasn't fixed points. Consequently it follows that $M=[a, b]$, where $a\leq b$ q. e. d.

DEFINITION 2. *The set M is compact if it is bounded and closed. The set M is nearly compact if either is a bounded, closed interval, i.e. has the form* $[a, b]$, *where* $a, b\in\mathbf{R}$, $a\leq b$ *or we can obtain it from* $[a, b]$ *by taking out at most countable, in pairs disjoint intervals of the form* (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$, *where* $\alpha, \beta\in\mathbf{R}$, $\alpha<\beta$, *such that the endpoints a, b remain in M.*

Observation. If we take from $[a, b]$ in pairs disjoint intervals of the form (α, β), $(\alpha, \beta]$, $[\alpha, \beta)$, where $\alpha, \beta\in\mathbf{R}$, $\alpha<\beta$, such that the endpoints a, b remain, then these intervals are at most countable, because in every taked interval we can choose a rational number, and we realise a one to one correspondence between the set of taken intervals and a subset of the rational numbers.

Examples. Every compact set is nearly compact. Indeed, after (1) every compact set looks either like a segment, i. e. is a bounded, closed interval, or we can obtain it from a segment by taking out at most countable, in pairs disjoint intervals of the form (α, β). Every finite set is compact, so it is nearly compact. The set $M=\{0\}\cup\left\{\frac{1}{n}\,|\,n\in\mathbf{N}^*\right\}$ is compact, so it is nearly compact. The classical Cantor set obtained from the $[0,1]$ interval is closed and perfect, so it is compact and nearly compact, too.

The set $M=\{0\}\cup\left(\bigcup_{n\geq1}\left(\frac{1}{2.n}, \frac{1}{2.n-1}\right)\right)$ isn't compact but it is nearly compact, because we take out from the segment $[0,1]$, for $n\geq1$ the intervals $\left(\frac{1}{2.n+1}, \frac{1}{2.n}\right]$. Similarly with the construction of the Cantor sets, we can build not only compact sets, but more, nearly compact sets like in the following: from the segment $[0,1]$ we take out one of the intervals: $\left(\frac{1}{3}, \frac{2}{3}\right)$ or $\left(\frac{1}{3}, \frac{2}{3}\right]$ or $\left[\frac{1}{3}, \frac{2}{3}\right)$. There remain the intervals $\left[0, \frac{1}{3}\right)$ or $\left[0, \frac{1}{3}\right]$, from which we take out one of the following intervals: $\left(\frac{1}{9}, \frac{2}{9}\right)$ or $\left(\frac{1}{9}, \frac{2}{9}\right]$ or $\left[\frac{1}{9}, \frac{2}{9}\right)$, and the intervals $\left[\frac{1}{3}, 1\right]$ or $\left(\frac{1}{3}, 1\right]$, respectively from which we take out one of the following intervals: $\left(\frac{7}{9}, \frac{8}{9}\right)$ or $\left(\frac{7}{9}, \frac{8}{9}\right]$ or $\left[\frac{7}{9}, \frac{8}{9}\right]$ etc. The remaining intervals are again divided into three parts, and the middle part is taken out in the form (α, β) or $[\alpha, \beta)$ or $(\alpha, \beta]$, and so on.

LEMMA. *The set* $M\subset\mathbf{R}$ *is a complete lattice if and only if M is nearly compact.*

Proof. We consider the restriction to M of the naturally ordering relation of the real line. In correspondence with this M is a lattice, because every pair of real numbers from M has trivially their minimum and maximum in M. M becomes a complete lattice, if every subset of elements of M has infimum and supremum in M.

Let us suppose that M is a complete lattice. If M is an unbounded set, then for the subset formed by a sequence with terms in M and with limit $-\infty$ or $+\infty$, there do not exist the infimum and the supremum in M, respectively. This means that M is bounded. Now, we consider $a\in\mathbf{R}$ the infimum and $b\in\mathbf{R}$ the supremum of the set M on the real axis. Here we mention the trivial fact, that for every bounded subset of elements from M their usual infimum and supremum exists on the real axis but we must take the infimum and the supremum in correspondence with the elements of the set M. The numbers a and b are the infimum and the supremum of the whole set M, so they belong to M. Now we must show that we can't take out from the segment determined by the infimum and the supremum endpoints of the set M a closed interval. Let us suppose the contrary. If x is a number between the infimum and supremum of the set M such that does not belong to M then we take out from M the biggest closed interval which contains x, denoted $[\alpha, \beta]$ with $\alpha\leq x\leq\beta$, $\alpha, \beta\in\mathbf{R}$. Then α is an upper, β is a below accumulation point for the set $M\cap(-\infty,\alpha)$ and $M\cap(\beta,+\infty)$ respectively, because $[\alpha, \beta]$ is the greatest, disjoint interval with M. But we find that the set $(-\infty,\alpha)\cap M$ has no supremum point in M, and the set $(\beta,+\infty)\cap M$ has no infimum point in M which means a contradiction.

Vice versa, let us suppose that M is a nearly compact set. For every subset of M, we denote by i the infimum on the real axis, which is a finite real number. If $i\in M$ then i is the infimum of the set $(i,+\infty)\cap M$ in correspondence with the set M. If $i\notin M$ then we consider the greatest, disjoint interval with M, which contains i. The form of this interval is $(\alpha, i]$, because M is nearly compact. Then $\alpha\in M$ and will be the infimum point for the subset $(\alpha,+\infty)\cap M=(i,+\infty)\cap M$ relatively to the set M. Similarly for the supremum point of every subset of M.

Now we give the following:

DEFINITION 3. The set M has the strong fixed point property if every monotone increasing function $f: M\to M$ has a fixed point.

THEOREM 2. *The set M is nearly compact if and only if M has the strong fixed point property.*

First proof. After [2] we will use the Tarski theorem: "a lattice M is complete if and only if every order-preserving map f of M to itself has a fixed point". After the previous lemma M is a complete lattice if and only if M is a nearly compact set. The role of the isotone functions on the lattice M is played by the monotone increasing functions.

Second proof. If M has the strong fixed point property, then we will show that it is nearly compact. First we will demonstrate that M is bounded. If the set M

is unbounded on the right, then by the definition for every $n \in N$ there exists $x \in M$ such that $x > n$. Because $M \neq \varnothing$, let $x_1 \in M$. We define the function $f: M \to M$ such that for every $x \in (-\infty, x_1) \cap M$, $f(x) = x_1$. But M is an upper unbounded set so there exists $x_2 \in M$ such that $x_2 > [x_1] + 1$. Now we define for every $x \in [x_1, x_2) \cap M$, $f(x) = x_2$ and so on. Inductively for the value $x_n \in M$ there exists the point $x_{n+1} \in M$ such that $x_{n+1} > [x_n] + 1$ and for every $x \in [x_n, x_{n+1}) \cap M$ we define $f(x) = x_{n+1}$. In this way we build the monotone increasing function f which has no fixed points. This contradiction assures us that M is upper bounded. In a similar way we can conclude that the set M is below bounded so M is a bounded set. This means that there exists $a \in \mathbf{R}$, the infimum and $b \in \mathbf{R}$, the supremum of the set M on the real line. It is true that $a, b \in M$ and they will be the infimum and the supremum, respectively, of the whole set M. For example if $a \notin M$ then there exists a strictly decreasing sequence $\{x_n\}_{n \in \mathbf{N}^*} \subset M$, such that $\lim_n x_n = a$. We construct the following function: if $x \in (x_1, +\infty) \cap M$ then we define $f(x) = x_1$, ..., if $x \in (x_{n+1}, x_n] \cap M$ then we put $f(x) = x_{n+1}, \ldots$. It is easy to see that f is monotone increasing and has not fixed points.

Let us consider an arbitrary, maximal length interval of the form $[\alpha, \beta]$, where $\alpha \leq \beta$, $\alpha, \beta \in \mathbf{R}$, in the complementary set of the set M. Then α is an upper accumulation point for the set M, so there exists a strictly increasing sequence $\{x_n\}_{n \in \mathbf{N}^*} \subset M$ with $\lim_n x_n = \alpha$, and β is a below accumulation point for the set M, so there exists a strictly decreasing sequence $\{x'_n\}_{n \in \mathbf{N}^*} \subset M$ with $\lim_n x'_n = \beta$. Now we are ready to construct the function $f: M \to M$ in the following way: for every $x \in (-\infty, x_1) \cap M$, $f(x) = x_1$, and for $x \in (x'_1, +\infty) \cap M$, $f(x) = x'_1$, and inductively for every $n \in N^*$ when $x \in [x_n, x_{n+1}) \cap M$, $f(x) = x_{n+1}$, and for $x \in (x'_{n+1}, x'_n] \cap M$, $f(x) = x'_{n+1}$, respectively. The function f is monotone increasing but hasn't fixed points. This means that our supposition is false, so we can not take out from the set M intervals of the form $[\alpha, \beta]$.

Conversely, let us suppose that M is a nearly compact set and we take the monotone increasing function $f: M \to M$. It is easy to see that M is a union of in pairs disjoint intervals with arbitrary form, at most continuum cardinality. It may happen that in the union there appear points which we can interpret like a closed interval $[\alpha, \alpha]$. Using the axiom of choice we can choose from every interval a fixed number which forms the index set I. This is a bounded, ordered set, too. Let us consider the family of intervals with the property: in every interval there exists a number x such that $f(x) \geq x$. This family is nonvoid because in the left interval there exists a number, exactly $a = \inf M$ such that $f(a) \geq a$. Let us denote by $J \subset I \subset M$ the index set of this family of intervals with the previous property. The fact that $\max I \in J$ means that there exists $x \in M$ in the interval of $\max I$, such that $f(x) \geq x$. But $f(b) \leq b$, so the Knaster theorem for the function $f_{/[x,b]}: [x, b] \to [x, b]$ assures us the existence of a fixed point. So there remains the case $\max I \notin J$.

If there exists $\max J \in J < \max I$, we consider the interval corresponding to this value. Let x be the value in this interval for $f(x) \geq x$, and α the greatest real number such that we have the intervals of the form $[x, \alpha]$ or $[x, \alpha)$. In the first case, if $f(\alpha) \leq \alpha$ then we apply the theorem of Knaster for $f_{/[x,\alpha]}: [x, \alpha] \to [x, \alpha]$. If $f(\alpha) > \alpha$, then $f(\alpha) \notin (x, \alpha]$, $f(\alpha) \in M$ is in an interval whose index is greater then $\max J$. The maximum condition is in contradiction with $f(f(\alpha)) \geq f(\alpha)$. In the second case, because M is a nearly compact set, we take out from the segment $[a, b]$ the interval of the form $[\alpha, \beta)$, so $\beta \in M$. The maximum condition for J implies that $f(\beta) < \beta$. But $x < \beta$, so $x \leq f(x) \leq f(\beta) < \alpha < \beta$ because $f(\beta) \in M$. We use the monotony of f, so $f(x) \leq f(f(x))$, $f(f(\beta)) \leq f(\beta)$, and the Knaster's theorem for f on $[f(x), f(\beta)]$ assures us the existence of a fixed point.

If $\max J$ does not exist then there exists the supremum of the set J in correspondence with the real line and we denote it by $\sup J$. But the definition of supremum J assures us the existence of the strictly increasing sequence $\{t_n\}_{n \in \mathbf{N}^*} \subset J \subset I \subset M$, with $\lim_n t_n = \sup J$, and for every t_n the existence of in pairs disjoint intervals. There are two possibilities: either $\sup J \in M$ or $\sup J \notin M$. In the first case M contains an interval like $[\sup J, \alpha)$ or $[\sup J, \alpha]$. Because $\sup J \notin J$, J contains no element of these intervals. This means that for every $x \in [\sup J, \alpha)$ or $x \in [\sup J, \alpha]$, $f(x) < x$ in particular $f(\sup J) < \sup J$. In the second case we take out from the segment $[a, b]$ the interval $[\sup J, \alpha)$. The supremum condition for J implies that $f(\alpha) < \alpha$. In both cases we obtain by the definition of the supremum of J, for enough great n that $f(\sup J) < t_n < t_{n+1} < \sup J$ and $f(\alpha) < t_n < t_{n+1} < \alpha$, respectively. But in the interval corresponding to t_{n+1} there exists v such that $f(v) \geq v$. So $f(v) \geq v > t_n > f(\sup J)$ and $f(v) \geq v > t_n > f(\alpha)$ respectively, with $v < \sup J$, $v < \alpha$. These are contrary to the fact that f is monotone increasing function.

Consequences:

1. *If M is a compact set then every monotone increasing function $f: M \to M$ has a fixed point.*

2. *If $M = [a, b] \subset R$, $a \leq b$ then every monotone increasing function $f: [a, b] \to [a, b]$ has a fixed point (theorem of Knaster). We mention, that the first proof of theorem 2 does not use Knaster's theorem.*

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