REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 25, N^{os} 1–2, 1996, pp. 111–119

APPROXIMATION OF CONTINUOUSLY GÂTEAUX DERIVABLE FUNCTIONALS BY MARKOV OPERATORS

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1. INTRODUCTION

Let (X,d) be a compact metric space, let $C(X)$ denote the Banach lattice of all real-valued continuous functions on X and let $L:C(X)\to C(X)$ be a positive linear operator.

Estimates of the errors $|L(f)(x)-f(x)|$ (*or* $\|L(f)-f\|$) involving among others the usual modulus of continuity defined by

$$\omega(f;\delta)=\sup\{|f(x)-f(y)|\ ;\ x,y\in X,\ d(x,y)\le\delta\},$$

are studied in several papers [13–21], [11], [6]. For $X=[a,b]$, $a,b\in\mathbf{R}$, such problems can be found for example, in [4].

The case of approximation of continuously differentiable functions on $[a,b]$ is also well-known (see e.g. [3–4], [6–8]).

When $[a,b]$ is replaced by X - a compact subset in a locally convex Hausdorff space, for functions $f\in C(X)$ having the so-called Mean Value Property (briefly MVP), extensions of this last case were obtained in [13], thus covering some results in [3].

However, the result in [13] do not cover the pointwise estimates in [4], [6–9], as for example the following easily obtained by combining Corolarry 2.2 and Theorem 2.3 in [9].

THEOREM 1.1 *Let* $L:C[a,b]\to C[a,b]$ *be a positive linear operator satisfying* $L(e_0)(x)=e_0(x)$, *where* $e_0(x)=1$, $\forall\, x\in[a,b]$. *For each* $f\in C^1[a,b]$ *we have*

$$|L(f)(x)-f(x)|\le\|f'\|\cdot|L(t-x)(x)|+2\cdot L(|t-x|)(x)\cdot$$
$$\cdot\omega\left[f';L(|t-x|^2)(x)/L(|t-x|)(x)\right],\quad\forall\ x\in[a,b],$$

where $L(t-x)(x), L(|t-x|)(x)$ *and* $L(|t-x|^2)(x)$ *mean that L is applied to* $t-x$, $|t-x|$ *and* $|t-x|^2$ *considered as function of t.*

The main purpose of this paper is to extend Theorem 1.1 to the case when $[a,b]$ is replaced by X – a compact convex subset of a linear normed space, for functions having continuous Gâteaux derivative on X. In case when $X=[a,b]$, our results are even refinements of Theorem 1.1.

On the other hand, since in general for Gâteaux derivable functionals in an abstract normed space, the MVP cannot be obtained in the form in [13], our results will be more general than those in [13].

Applications to Bernstein-Lototsky-Schnabl operator are given.

2. PRELIMINARIES

Let $(E,\|\cdot\|_1),(F,\|\cdot\|_2)$ be two real normed spaces, $X\subset E$ be a compact convex subset of E and let $f:X\to F$.

DEFINITION 2.1. *The modulus of continuity of f on X with step* $\delta\ge 0$ *is defined by*

$$\omega(f;\delta)=\sup\{\|f(x)-f(y)\|_2\ ;\ x,y\in X, \|x-y\|_1\le\delta\}, \delta\in[0,d(X)],$$

where $d(X)=\sup\{\|x-y\|_1; x,y\in X\}<+\infty$ *is the diameter of the compact set X.*

Also, the least concave majorant of $\omega(f;\delta)$ is defined by $\tilde{\omega}(f;\delta)=$

$$=\sup\left\{\sum_{i=1}^{n}\lambda_i\cdot\omega(f;\delta_i)\ ;\ n\in\mathbf{N},\sum_{i=1}^{n}\lambda_i=1,\sum_{i=1}^{n}\lambda_i\delta_i=\delta,\ \lambda_i\ge 0\right\}, \text{if } \delta\in[0,d(X)] \text{ and}$$

$$\tilde{\omega}(f;\delta)=\omega(f;d(X)),\ \text{ if }\ \delta>d(X)$$

(see e.g. [6] in the case when F is the real axis).

LEMMA 2.2. (i) *For all* $\delta_1,\delta_2\in[0,d(X)]$ *with* $\delta_1+\delta_2\in[0,d(X)]$ *we have*

$$\omega(f;\delta_1+\delta_2)\le\omega(f;\delta_1)+\omega(f;\delta_2).$$

(ii) *For all* $\delta\ge 0$ *we have*

$$\omega(f;\delta)\le\tilde{\omega}(f;\delta)\le 2\omega(f;\delta).$$

Proof. (i) We will reason as in the case of real functions. Let x, y be such that $\|x-y\|_1\le\delta_1+\delta_2$ and write $z=\alpha x+(1-\alpha)y\in X$, where $\alpha=\delta_2/(\delta_1+\delta_2)$. We have:

$$\|x-z\|_1=\|(1-\alpha)(x-y)\|_1=(1-\alpha)\cdot\|x-y\|_1\le(1-\alpha)(\delta_1+\delta_2)=\delta_1$$

and

$$\|y-z\|_1=\|\alpha(y-x)\|_1=\alpha\|y-x\|_1\le\delta_2.$$

Hence we obtain

$$\|f(x)-f(y)\|_2\le\|f(x)-f(z)\|_2+\|f(z)-f(y)\|_2\le\omega(f;\|x-z\|_1)+$$
$$+\omega(f;\|z-y\|_1)\le\omega(f;\delta_1)+\omega(f;\delta_2),$$

and passing to supremum with $x,y\in X$ we get

$$\omega(f;\delta_1+\delta_2)\le\omega(f;\delta_1)+\omega(f;\delta_2).$$

(ii) An immediate consequence of (i) is

$$(1)\qquad \omega(f;\lambda\delta)\le(1+\lambda)\cdot\omega(f;\delta),\qquad \forall\ \lambda,\delta\ge 0.$$

Now, the inequality $\omega(f;\delta)\le\tilde{\omega}(f;\delta)$ is obvious by definition. Also, by (1) we obtain

$$\sum_{i=1}^{n}\lambda_i\cdot\omega(f;\delta_i)=\sum_{i=1}^{n}\lambda_i\cdot\omega(f;(\delta_i/\delta)\delta)\le\sum_{i=1}^{n}\lambda_i(1+\delta_i/\delta)\cdot\omega(f;\delta)=$$
$$=\left(\sum_{i=1}^{n}\lambda_i+\sum_{i=1}^{n}\lambda_i\delta_i/\delta\right)\cdot\omega(f;\delta)=2\cdot\omega(f;\delta).$$

Passing now to supremum, we immediately get $\tilde{\omega}(f;\delta)\le 2\cdot\omega(f;\delta)$.

It is known

DEFINITION 2.3. *Let* $f:E\to R$ *be a functional,* $X\subset E$ *and* $x\in X$. *We say that f is Gâteaux derivable at x if for all* $h\in E$, *there exists the limit* $\lim_{t\to 0}[f(x+th)-f(x)]/t=f'_x(h)$ *and* f'_x *is linear and continuous as function of h, i.e.* $f'_x\in E^*$.

Also, f is called Gâteaux derivable on X if $f'_x\in E^*$ for all $x\in X$ and in this case we can define $f':X\to E^*$ by $f'(x)=f'_x$.

The proofs of our main results also require the following known Jessen's inequality.

THEOREM 2.4 *(see e.g.* [2], [10]). *Let (X, d) be a compact metric space, $G \in C[m, M]$ a concave function on the interval [m, M] and let $A: C(X) \to R$ be a positive linear functional satisfying*

$$A(1_X) = 1 \text{ and } f \geq 0 \text{ implies } A(f) \geq 0 .$$

For any continuous function $g: X \to [m, M]$ we have $A(G(g)) \leq G(A(g))$.

3. MAIN RESULTS

Let $(E, \|\cdot\|)$ be a real normed space and let $X \subset E$ be a compact convex subset. Write

$$D^1_X(E) = \{f: E \to R ; f \text{ is Gâteaux derivable on } X \text{ and } f': X \to E^* \text{ is continuous on } X\},$$

and

$$C^1(X) = \{f|_X ; f \in D^1_X(E)\},$$

where $f|_X$ means the restriction of f on X.

Remark. If $f \in C^1(X)$ then by e.g. [5,p. 341] we get that f is Fréchet differentiable on X, which implies (see e.g. [5,p. 340]) that $f \in C(X)$, i.e. $C^1(X) \subset C(X)$.

We say that $L: C(X) \to C(X)$ is a Markov operator on $C(X)$ if it is positive, linear and satisfies $L(1_X) = 1_X$.

The first main result of this paper is

THEOREM 3.1. *Let X be a compact convex subset of E and let $L: C(X) \to C(X)$ be a Markov operator on C(X). For each $f \in C^1(X)$ we have*

$$(2) \quad |L(f)(x) - f(x)| \leq |L[f'_x\,(t-x)](x)| + 2 \cdot L(\|t-x\|)(x) \cdot \omega\left[f'; L(\|t-x\|^2)(x) / L(\|t-x\|)(x)\right], \quad \text{for all } x \in X,$$

where $f'_x \in E^*$ and E^* is endowed with the usual norm

$$|||x^*||| = \sup\{|x^*(x)|;\ x \in E,\ \|x\| \leq 1\}, \quad \forall x^* \in E^*.$$

Proof. By the mean value theorem (see e.g. [5,p. 323]) we have

$$f(t) - f(x) = f'_x(t-x) + f'_{x+\tau(t-x)}(t-x) - f'_x(t-x), \quad x, t \in X,$$

where τ exists and belongs to (0,1).

Applying L we immediately get

$$|L(f)(x) - f(x)| \leq |L[f'_x\,(t-x)](x)| + \left|L\left(\left[f'_{x+\tau(t-x)} - f'_x\right](t-x)\right)(x)\right| \leq$$

$$\leq |L[f'_x\,(t-x)](x)| + L\left(\left|\left[f'_{x+\tau(t-x)} - f'_x\right](t-x)\right|\right)(x) \leq |L[f'_x\,(t-x)](x)| +$$

$$+ L\left(|||f'_{x+\tau(t-x)} - f'_x||| \cdot \|t-x\|\right)(x) \leq |L[f'_x\,(t-x)](x)| +$$

$$+ L\left(\|t-x\| \cdot \omega\left(f'; \|x+\tau(t-x)-x\|\right)\right)(x) \leq |L[f'_x\,(t-x)](x)| + L\left[\|t-x\| \cdot \omega\left(f'; \|t-x\|\right)\right](x),$$

where L is applied to $f'_x\,(t-x)$ and to $\|t-x\| \cdot \omega(f'; \|t-x\|)$, considered as functions of t, with fixed x.

Therefore we can write

$$(3) \quad |L(f)(x) - f(x)| \leq |L[f'_x\,(t-x)](x)| + L[\|t-x\| \cdot \tilde{\omega}(f'; \|t-x\|)](x).$$

For $x \in X$ we have two possibilities:

(i) $f(x) = L(f)(x)$; (ii) $f(x) \neq L(f)(x)$.

In the case (i), obviously (2) holds. In the case (ii) we have $L(\|t-x\|)(x) > 0$. Indeed, let us suppose that in this case $L(\|t-x\|)(x) = 0$. By the mean value theorem we obtain

$$|f(t) - f(x)| \leq |||f'_{x+\tau(t-x)}||| \cdot \|t-x\| \leq M \cdot \|t-x\|,$$

where $M = \sup\{|||f'_x|||; x \in X\} < +\infty$, since f' is continuous on the compact set X.

Applying L we get

$$|L(f)(x) - f(x)| \leq L(|f(t) - f(x)|)(x) \leq M \cdot L(\|t-x\|)(x) = 0,$$

which implies the contradiction $L(f)(x) = f(x)$.

Now, for $x \in X$ with $f(x) \neq L(f)(x)$, let us define

$$A_x: C(X) \to R \text{ by } A_x(f) = L(\|t-x\| \cdot f(t))(x) / L(\|t-x\|)(x)$$

and

$$g_x : X \to [0, d(X)] \text{ by } g_x(t) = \|t - x\|.$$

Since $\tilde{\omega}$ is concave, by Theorem 2.4 we get

$$A_x(\tilde{\omega}(f'; g_x)) \le \tilde{\omega}(f'; A_x(g_x)),$$

that is

(4) $$L[\|t - x\|\tilde{\omega}(f'; \|t - x\|)](x) / L(\|t - x\|)(x) \le$$
$$\le \tilde{\omega}\left[f'; L(\|t - x\|^2)(x) / L(\|t - x\|)(x)\right],$$

which together with (3) and with Lemma 2.2, (ii), proves the theorem.

Remark. Let E be the real axis and $X=[a,b]$. Then f'_x becomes the usual derivatives $f'(x)$ and $f'_x\ (t - x)$ becomes $f'(x) \cdot (t - x)$. Hence we get

$$|L[f'_x\ (t - x)](x)| = |L[f'(x)(t - x)](x)| = |f'(x)| \cdot |L(t - x)(x)|,$$

and by Theorem 3.1 we obtain

$$|L(f)(x) - f(x)| \le |f'(x)| \cdot |L(t - x)(x)| + 2 \cdot L(|t - x|)(x) \cdot$$
$$\cdot \omega\left[f'; L(|t - x|^2)(x) / L(|t - x|)(x)\right], \quad \forall\ x \in [a, b],$$

which because of the term $|f'(x)| \cdot |L(t - x)(x)|$ is obviously a refinement of Theorem 1.1.

COROLLARY 3.2. *In the conditions of Theorem 3.1.we have*

$$|L(f)(x) - f(x)| \le |L[f'_x\ (t - x)](x)| + 2 \cdot \left[L(\|t - x\|^2)(x)\right]^{1/2} \cdot$$
$$\cdot \omega\left[f'; \left[L(\|t - x\|^2)(x)\right]^{1/2}\right], \ \forall\ x \in X.$$

Proof. Since $\tilde{\omega}(f'; \delta)$ is concave, by e.g. [12,p. 44], $\tilde{\omega}(f'; \delta) / \delta$ decreases as function of $\delta > 0$ which immediately give

$$\tilde{\omega}(f'; \lambda\delta) / (\lambda\delta) \le \tilde{\omega}(f'; \delta) / \delta, \ \forall\ \lambda \ge 1, \ \ \delta > 0,$$

that is

(5) $$\tilde{\omega}(f'; \lambda\delta) \le \lambda\tilde{\omega}(f'; \delta), \ \ \forall\ \lambda \ge 1, \ \ \delta > 0.$$

Now, taking into account that

$$1 \le \left[L(\|t - x\|^2)(x)\right]^{1/2} / L(\|t - x\|)(x),$$

by (4) and (5) we immediately obtain

$$L[\|t - x\| \cdot \tilde{\omega}(f'; \|t - x\|)](x) \le \left[L(\|t - x\|^2)(x)\right]^{1/2} \tilde{\omega}\left(f'; \left[L(\|t - x\|^2)(x)\right]^{1/2}\right),$$

which together with (3) and with Lemma 2.2,(ii), proves the corollary.

Remark. By (4) and by (3) we immediately obtain that if $f \in C^1(X)$ has $\omega(f'; \delta)$ concave as function of δ, then

$$|L(f)(x) - f(x)| \le |L[f'_x\ (t - x)](x)| + L(\|t - x\|)(x) \cdot$$
$$\cdot \omega\left[f'; L(\|t - x\|^2)(x) / L(\|t - x\|)(x)\right].$$

COROLLARY 3.3. *Let $(E, <\cdot,\cdot>)$ be a real Hilbert space. In the conditions of Theorem 3.1 we have*

$$|L(f)(x) - f(x)| \le |L(< y'_x, t >)(x) - < y'_x, x >| + 2 \cdot L(\|t - x\|)(x) \cdot$$
$$\cdot \omega\left[f'; L(\|t - x\|^2)(x) / L(\|t - x\|)(x)\right], \quad \forall\ x \in X,$$

where $y'_x \in X$ is such that $\|y'_x\| = \|f'_x\|$ and $L(< y'_x, t >)(x)$ means that L is applied to $< y'_x, t >$ as function of t.

Proof. By the well-known Riesz's result, there exists (unique) $y'_x \in X$ such that

$$\|y'_x\| = \|f'_x\| \text{ and } f'_x\ (t - x) = < y'_x, t - x > = < y'_x, t > - < y'_x, x >.$$

Then our corollary is an immediate consequence of Theorem 3.1.

4. APPLICATIONS

We will apply the previous results to the case of Bernstein-Lototsky-Schnabl operator. Let $(E, <\cdot,\cdot>)$ be a real Hilbert space and $X \subset E$ a compact convex subset.

Keeping the notation, define as in [16,p. 454-455] the n-th Bernstein-Lototsky-Schnabl operator with respect to $\mathcal{U}^{(V)}$, ρ, P and $\mathcal{Y}=\{y_x; x\in X\}$, by

$$B_n(f)(x) = B_{n,P,\rho}^{(\mathcal{U}(V),\mathcal{Y})} = \int_{X^n} f \circ \pi_{n,P} \cdot dx_{1\leq j\leq n} \nu_{x,j,\rho}^{(V,\mathcal{Y})}$$

and let us consider for $\mathcal{Y}$ that $y_x=x$, for all $x\in X$.

By Lemma 6,(i), in the same paper [16] we get

$$B_n(< y'_x , t >)(x) =< y'_x , x >, \forall \ x \in X,$$

where $B_n(< y'_x , t >)(x)$ means that B_n is applied to $g(t) =< y'_x , t >,\ t \in X$. Then by Corollary 3.2 we easily obtain:

COROLLARY 4.1 *For all* $f \in C^1(X)$ *we have*

$$|B_n(f)(x) - f(x)| \leq 2 \cdot \left[B_n\left(\|t - x\|^2\right)(x)\right]^{1/2} \cdot \omega\left(f';\left[B_n\left(\|t - x\|^2\right)(x)\right]^{1/2}\right), \quad x \in X.$$

Remark. Let us suppose that $f' : X \to E^*$ satisfies

$$\| f'_x - f'_y \| \leq M \cdot \|x - y\|^{\alpha}, \quad \forall \ x, y \in X, \text{ with fixed } \alpha \in (0,1].$$

Then obvioulsy $\omega(f';\delta) \leq M \cdot \delta^{\alpha}$ and the previous estimate becomes

$$|B_n(f)(x) - f(x)| \leq 2 \cdot \left[B_n\left(\|t - x\|^2\right)(x)\right]^{(\alpha+1)/2}, \forall \ x \in X.$$

Obvioulsly, this estimate cannot be obtained from the estimates in [1, Corollary 2 and 3].

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Received 15.06.1993

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