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ON SOME GRONWALL-TYPE INEQUALITIES FOR MONOTONIC OPERATORS

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1. INTRODUCTION

The integral inequalities play a fundamental part within the study of the existence, uniqueness, stability, boundedness, continuability (and other qualitative aspects) of the solutions of differential and integral equations. In [5] there were established operatorial inequalities of the type Gronwall and Bihari for increasing and decreasing monotonic operators. In [3] there was established an inequality in the case of an increasing operator. This result is based on Theorem 2 from [5], which is reminded further down:

Let X be a Banach space, and let K be a semiordered cone; $x \geq y$ means $x - y \in K$.

Consider the inequation $u \leq Au + f$ where A is a monotonically decreasing positive operator. Suppose that the following conditions are fulfilled:

(i) Equation $y=Ay+f$ has the unique solution y^*, the limit of the sequence (y_n) defined by $y_{n+1} = Ay_n + f$.

(ii) There exists an element $u_0 \in X$ which verifies the inequalities $u_0 \leq Au_0 + f \equiv u_1$, $u_0 \leq Au_1 + f$.

Then $u_0 \leq y^*$.

Remark. If the nonnegative function $u_0(t)$, verifies the inequality

$$u_0(t) \leq c + \int_0^t a(s)V_1[u_0(s)]\mathrm{d}s \equiv u_1(t), \quad t \in [a,b] \tag{1}$$

$$u_0(t) \leq c + \int_0^t a(s)V_1[u_1(s)]\mathrm{d}s \tag{2}$$

where $V_1(y)$ is nonnegative, monotonically decreasing, and locally Lipschitzian, $a(s)$ nonnegative and continuous, then $u_0(t)$ verifies the inequalitiy

$$u_0(t) \le F_1^{-1}[F(t) + F_1(c)], \quad t \in [0,b], \tag{3}$$

where F_1 is the primitive of the function $1/V_1(y)$, $F_1^{-1}(y)$ is its inverse, while $F(t)$ is the primitive of $a(s)$.

In [3]–[4] there was established a Riccati-type inequality in the case of an increasing operator. We shall establish an analogous result in the case of a decreasing operator. Of course, the result will be obtained under supplementary conditions.

2. INEQUALITIES FOR DECREASING OPERATORS

THEOREM 1. *Let* $u_0(t) \in C[0,b]$, $a(t) \in C[0,b]$, $a(t) \ge 0$ *for any* $t \in [0,b]$ *and* $p,q,r \in \mathbf{R}_+^*$, $(q^2 \le 4pr)$. *If*

$$u_0(t) \le c + \int_0^t [pa(s)u_0^2(s) + qa(s)u_0(s) + ra(s)]\mathrm{d}s \equiv u_1(t) \tag{4}$$

$$u_0(t) \le c + \int_0^t [pa(s)u_1^2(s) + qa(s)u_1(s) + ra(s)]\mathrm{d}s \tag{5}$$

where $c > 0$, *and if* y_1 *is a particular solution of the equation*

$$y' = pa(t)y^2 + qa(t)y + ra(t), \tag{6}$$

then $u_0(t)$ *verifies the inequality* $u_0(t) \le y^*(t)$, *where*

$$y^*(t) = \exp\left[\int_0^t (2pa(s)y_1(s) + qa(s))\mathrm{d}s\right] \times \left[c - \int_0^t 2pa(s)\exp\left(\int_0^s (2pa(z)y_1(z) + qa(z))\mathrm{d}z\right)\mathrm{d}s\right]^{-1}. \tag{7}$$

Proof. Define the operator A by

$$Au = \int_0^t a(s)V(u(s))\mathrm{d}s, \quad t \in [0,b] \tag{8}$$

where

$$V(u(t)) = pu^2(t) + qu(t) + r.$$

If $y^*(t)$ is solution of the equation

$$y(t) = c + \int_0^t [pa(s)y^2(s) + qa(s)y(s) + ra(s)]\mathrm{d}s, \quad c > 0 \tag{9}$$

then we have

$$y^*(t) = \exp\left[\int_0^t (2pa(s)y_1(s) + qa(s))\mathrm{d}s\right] \times \left[c - \int_0^t 2pa(s)\exp\left(\int_0^s (2pa(z)y_1(z) + qa(z))\mathrm{d}z\right)\mathrm{d}s\right]^{-1} \tag{10}$$

where y_1 verifies equation (6). Therefore it results

$$u_0(t) \le y^*(t). \tag{11}$$

THEOREM 2. *Let* $v, w_0 \in C[\mathbf{R}_+^2, \mathbf{R}_+]$ *and* $c \ge 0$. *If the function* $w_0(x,y)$ *verifies the inequalities*

$$w_0(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_0(s,t)\mathrm{d}s\mathrm{d}t \equiv w_1(x,y), \quad x \ge x_0,\ y \ge y_0 \tag{12}$$

$$w_0(x,y) \le c + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_1(s,t)\mathrm{d}s\mathrm{d}t \tag{13}$$

while operator $Aw_0(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_0(s,t)\mathrm{d}s\mathrm{d}t$, $x \ge x_0, y \ge y_0$, *is monotonically decreasing, then*

$$w_0(x,y) \le u^*(x,y) \tag{14}$$

where $u^*(x,y)$ *is solution of the equation*

$$u_x(x,y) = \left(\int_{y_0}^{y} v(x,t)\mathrm{d}t\right)u(x,y). \tag{15}$$

Proof. By (12) we easily obtain

$$w_1(x,y) = C + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_0(s,t)\,\mathrm{d}s\mathrm{d}t \text{ and } w_0(x,y) \le w_1(x,y) \text{ then}$$

$$w_{1x} = \int_{y_0}^{y} v(x,t)w_0(x,t)\mathrm{d}t \le \int_{y_0}^{y} v(x,t)w_1(x,t)\mathrm{d}t$$

or

(16) $$w_{1x} \le \left(\int_{y_0}^{y} v(x,t)\mathrm{d}t\right) w_1(x,y)$$

From the comparison theorem [1] it results $w_1(x,y) \le u^*(x,y)$, from which we obviously have $w_0(x,y) \le u^*(x,y)$, too.

Since $u^*(x,y) = c\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,\mathrm{d}s\,\mathrm{d}t\right)$, it results Wendorff's inequality [2], hence

(17) $$w_0(x,y) \le c\exp\left(\int_{x_0}^{x}\int_{y_0}^{y} v(s,t)\,\mathrm{d}s\,\mathrm{d}t\right)$$

THEOREM 3. *Let the functions* $v, w_0, h \in C[\mathbf{R}_+^2, \mathbf{R}_+]$; *if*

(i) $w_0(x,y)$ *verifies the inequalities*

(18) $$w_0(x,y) \le h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_0(s,t)\,\mathrm{d}s\mathrm{d}t \equiv w_1(x,y),\quad x \ge x_0,\ y \ge y_0$$

(19) $$w_0(x,y) \le h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_1(s,t)\,\mathrm{d}s\mathrm{d}t$$

(ii) *operator* $Aw_0(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)w_0(s,t)\,\mathrm{d}s\mathrm{d}t,\ x \ge x_0,\ y \ge y_0,$

is monotonically decreasing, then

(20) $$w_0(x,y) \le u^*(x,y)$$

where $u^*(x,y)$ *is solution of the equation*

$$u_x(x,y) = \left(\int_{y_0}^{y} v(x,t)\mathrm{d}t\right) u(x,y) + \int_{y_0}^{y} v(x,t)h(x,t)\mathrm{d}t.$$

Proof. One proceeds as in Theorem 2, using also the comparison theorem [1]. In this case $u^*(x,y)$ is

$$u^*(x,y) = h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t)h(s,t)\exp\left(\int_{s}^{x}\int_{t}^{y} v(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta\right)\mathrm{d}s\,\mathrm{d}t.$$

Wendorff's inequality [2] is obtained in this case, too.

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