REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 25, N^{os} 1–2, 1996, pp.127–136

EQUIVALENCE CLASSES IN THE SET OF EFFICIENT SOLUTIONS

LIANA LUPŞA, DOREL I. DUCA, EUGENIA DUCA
(Cluj-Napoca)

Let X be a nonvoid set of $\mathbf{I}^p$, where by $\mathbf{I}$ we denote the set of integer numbers, let $f = (f_1,...,f_p): X \to \mathbf{I}^p$ and let $s: X \to \mathbf{I}$, $s = f_1 + \ldots + f_p$.

In the following, we denote by v-min $(f; X)$ the vectorial optimization problem which possesses the constraint set X and the objective functions f_j, $j \in \{1, \ldots, p\}$.

DEFINITION 1. *A point* $x^0 \in X$ *is said to be a min-efficient solution for* v-min $(f; X)$ *if there is no* $x \in X$ *such that:*

(1) $$f_j(x) \le f_j(x^0) \text{ for each } j \in \{1, \ldots, p\}$$

with at least one strict inequality.

Remark 1. Because $s(X) \subseteq \mathbf{I}$, a point $x \in X$ is a min-efficient solution for problem v-min $(f; X)$ if and only if there is no $y \in X$ such that

(2) $$f_j(y) \le f_j(x), \text{ for each } j \in \{1, \ldots, p\}$$

and

(3) $$s(y) \le s(x) - 1.$$

Let min-*EF* $(f; X)$ be the set of min-efficient solutions for problem v-min $(f; X)$. In the set min-*EF* $(f; X)$ we introduce the following equivalence relation : if x, y are in min-*EF* $(f; X)$, we say that x is equivalent with y if

(4) $$f(x) = f(y).$$

If $t = (t_1, \ldots, t_p) \in \mathbf{R}^p$, then we agree that we denote by $|t|$ the real number defined by $|t| = t_1 + \ldots + t_p$.

Let (P) be the following parametric programming problem:

$$(P)\qquad \begin{cases} s(x) \to \min \\ f_j(x) \le t_j, \quad j \in \{1,\dots,p\} \\ x \in X \end{cases}$$

with $t \in \mathbf{R}^p$. If $t^0 \in \mathbf{R}^p$, then

i) by $P(t = t^0)$ we denote the problem

$$\begin{cases} s(x) \to \min \\ f_j(x) \le t_j^0, \quad j \in \{1,\dots,p\}. \\ x \in X \end{cases}$$

ii) by $SA(t = t^0)$ we denote the set of admissible solutions for $P(t = t^0)$,

$$(5)\qquad SA(t = t^0) = \{x \in X : f_j(x) \le t_j^0,\ j \in \{1,\dots,p\}\},$$

iii) by $SO(t = t^0)$ we denote the set of optimal solutions for $P(t = t^0)$.

Let $T_0 = \{t^0 \in \mathbf{I}^p : SO(t = t^0) \ne \varnothing\}$.

THEOREM 1. *If X is bounded, then a point $u \in X$ is a min-efficient solution for problem* v-min $(f; X)$ *if and only if there is $\alpha \in T_0$ such that*

$$(6)\qquad u \in P(t = \alpha) \text{ and } s(u) = |\alpha|.$$

Proof. Necessity. Let $u \in X$ be a min-efficient solution for problem v-min $(f; X)$. Taking

$$(7)\qquad \alpha = (f_1(u),\dots,f_p(u)) \in \mathbf{I}^p,$$

it is obvious that $u \in SA(t = \alpha)$. Because X is a bounded set, it is also a finite set. Then $SA(t = \alpha) \ne \varnothing$ implies that $SO(t = \alpha) \ne \varnothing$. Hence $\alpha \in T_0$. If $u \notin SO(t = \alpha)$, then there is $v \in X$ such that

$$f_j(v) \le \alpha_j \le f_j(u) \quad \text{for all} \quad j \in \{1,\dots,p\},$$

and

$$(8)\qquad s(v) < s(u).$$

Since $s(u) \in \mathbf{I}$, $s(v) \in \mathbf{I}$, (8) is equivalent with

$$s(v) \le s(u) - 1.$$

In view of remark 1, u is not an efficient solution for v-min $(f; X)$; that contradicts the assumption. Therefore u is an optimal solution for problem $P(t = \alpha)$ and, from (7), we get

$$s(u) = f_1(u) + \dots + f_p(u) = \alpha_1 + \dots + \alpha_p = |\alpha|.$$

Hence the condition is necessary.

Sufficiency. Let $u \in X$ having the property that there is $\beta \in T_0$ such that $u \in SO(t = \beta)$ and $s(u) = |\beta|$. If we suppose that u is not an efficient solution for v-min $(f; X)$, then, in view of remark 1, there exists $v \in X$ such that

$$(9)\qquad f_j(v) \le f_j(u),\ j \in \{1,\dots,p\}$$

and

$$(10)\qquad s(v) \le s(u) - 1.$$

But, because u is an optimal solution for $P(t = \beta)$, we have

$$(11)\qquad f_j(u) \le \beta_j,\ j \in \{1,\dots,p\}.$$

Since $v \in X$, from (9) and (11) it results that $v \in SA(t = \beta)$. Then (10) implies $u \notin SO(t = \beta)$; that contradicts the assumption $u \in SO(t = \beta)$. Hence u is a min-efficient solution for problem v-min $(f; X)$.*

Because X is a finite set and $f(X) \subseteq \mathbf{I}^p$, there are $a = (a_1,\dots,a_p) \in \mathbf{I}^p$ and $b = (b_1,\dots,b_p) \in \mathbf{I}^p$ such that

$$(12)\qquad a_j = \min\{f_j(x) : x \in X\},\ b_j = \max\{f_j(x) : x \in X\}$$

for all $j \in \{1,\dots,p\}$.

Let $T = \underset{j=1}{\overset{p}{\times}} [a_j, b_j]$.

LEMMA 2. *If $\alpha \in \mathbf{I}^p \setminus T$, then one and only one of following assertions is true:*

(i) $SA(t = \alpha) = \varnothing$;

(ii) $SO(t = \alpha) \ne \varnothing$, *and for each* $x^0 \in SO(t = \alpha)$ *we have* $s(x^0) \ne |\alpha|$.

Proof. We can be in one and only one of the two cases.

i) There is $k \in \{1,\dots,p\}$ such that

$$(13)\qquad \alpha_k < a_k.$$

Then by (12) we get that there is none $x \in X$ such that $f_k(x) \le \alpha_k$. Therefore $SA(t=\alpha)=\varnothing$.

ii) $a_j \le \alpha_j$ for each $j \in \{1,\dots,p\}$, and there is $k \in \{1,\dots,p\}$ such that

(14) $$b_k < \alpha_k.$$

If $SA(t=\alpha)=\varnothing$, then the assertion (i) is true. If $SA(t=\alpha)\ne\varnothing$, then, because X is a nonempty finite set, it follows that $SO(t=\alpha)\ne\varnothing$. Let $x^0 \in SO(t=\alpha)$. We have

(15) $$f_j(x^0) \le \alpha_j,\ (\forall)\ j \in \{1,\dots,p\}.$$

Then from (12), (14) and (15) we get that

$$f_j(x^0) \le \alpha_j,\ (\forall)\ j \in \{1,\dots,p\}\setminus\{k\} \text{ and } f_k(x^0) < \alpha_k.$$

That implies that $s(x^0)=\sum_{\substack{j=1\\ j\ne k}}^{p} f_j(x^0)+f_k(x^0) < \sum_{j=1}^{p}\alpha_j = |\alpha|$. Hence $s(x^0)\ne|\alpha|$.∗

In the following, we join for each $\alpha \in T$ the set $V(\alpha)$ defined by

(16) $$V(\alpha)=\mathop{\times}_{j=1}^{p}[a_j,\alpha_j].$$

Obvious we have:

LEMMA 3. *If* $\alpha \in T$, $\beta \in T$ and $\alpha \le \beta$, then $SA(t=\alpha)\subseteq SA(t=\beta)$.

This lemma has two important consequences.

COROLLARY 4. *If* $\beta \in T$ *and* $SA(t=\beta)=\varnothing$, *then* $SA(t=\alpha)=\varnothing$ *for each* $\alpha \in V(\beta)$.

Let $\beta \in T$ and $x \in SA(t=\beta)$. We denote by $U(f(x),\beta)$ the set

(17) $$U(f(x),\beta)=\mathop{\times}_{j=1}^{p}[f_j(x),\beta_j].$$

COROLLARY 5. *If* $\beta \in T$ *and* $x^0 \in SO(t=\beta)$, *then* $x^0 \in SO(t=\alpha)$ *for each* $\alpha \in U(f(x^0),\beta)$.

Proof. Let $\alpha \in U(f(x^0),\beta)$. Then we get $\alpha \le \beta$ and applying lemma 3 it follows $SA(t=\alpha)\subseteq SA(t=\beta)$. That implies that

(18) $$\min\{f(x): x\in SA(t=\alpha)\} \ge \min\{f(x): x \in SA(t=\beta)\} = s(x^0).$$

Because $\alpha \in U(f(x^0),\beta)$, we have

$$f_j(x^0)\le \alpha_j,\ (\forall)\ j\in\{1,\dots,p\}.$$

Hence

(19) $$x^0 \in SA(t=\alpha).$$

From (18) and (19) it results that $x^0 \in SO(t=\alpha)$.∗

Let $\alpha \in T$. We denote

$$T_0(t=\alpha)=\mathop{\times}_{j=1}^{p}[\alpha_j,b_j],$$

$$T_1(t=\alpha)=\begin{cases}[a_1,\alpha_1]\times\left(\mathop{\times}_{j=2}^{p}[\alpha_j,b_j]\right), & a_1<\alpha_1\\ \varnothing, & a_1=\alpha_1\end{cases}$$

$$T_p(t=\alpha)=\begin{cases}\mathop{\times}_{j=1}^{p-1}[a_j,b_j]\times[a_p,\alpha_p], & a_p<\alpha_p\\ \varnothing, & a_p=\alpha_p\end{cases}$$

and, if $p>2$,

$$T_i(t=\alpha)=\begin{cases}\mathop{\times}_{j=1}^{i-1}[a_j,b_j]\times[a_i,\alpha_i]\times\left[\mathop{\times}_{j=i+1}^{p}[\alpha_j,b_j]\right], & a_i<\alpha_i\\ \varnothing, & a_i=\alpha_i\end{cases}$$

for $i\in\{2,\dots,p-1\}$.

LEMMA 6. *If* $\alpha \in T$, *then*

(20) $$T=\bigcup_{i=0}^{p}T_i.$$

Proof. Because $T_j(t=\alpha)\subseteq T$ for each $j\in\{0,1,\dots,p\}$, we get

(21)
$$\bigcup_{i=0}^{p} T_i \subseteq T.$$

Let $t \in T$. Two cases are possible.

i) For each $j \in \{1,\ldots,p\}$ we have $t_j \geq \alpha_j$. Then $t_j \in [\alpha_j, b_j]$, $(\forall)$ $j \in \{1,..,p\}$. It follows that $t \in T_0(t = \alpha)$.

ii) There is $j \in \{1,\ldots,p\}$ such that $t_j < \alpha_j$. Let

(22)
$$k = \max\{j \in \{1,\ldots,p\}:\ t_j < \alpha_j\}.$$

We shall prove that $t \in T_k$.

For each $j \in \{1,\ldots,p\}$ we can be in one and only one of the three cases:

a) $j \in \{1,\ldots,k-1\}$ and then evidently $t_j \in [a_j, b_j]$;

b) $j = k$ and then $t_k \in [a_j, \alpha_j]$;

c) $j \in \{k+1,\ldots,p\}$ and then, from (22) it follows that $t_j \geq \alpha_j$.

Hence $t \in T_k(t = \alpha)$. Because t is arbitrary choosen in T, we get that

(23)
$$T \subseteq \bigcup_{i=0}^{p} T_i(t = \alpha).$$

From (21) and (23) it follows (20).*

Using the conclusion of lemma 6 we give an algorithm for finding all equivalence classes for a vector optimization problem with integer variables.

DESCRIPTION OF THE ALGORITHM

Step 0. Put $i := 1$, $h := 1$, $j := 0$, $T_1 := T$.

Step 1. For each $k \in \{1,\ldots,p\}$ put

$$u_k = \min\{t_k : t = (t_1,\ldots,t_p) \in T_i\},\ v_k = \max\{t_k : t = (t_1,\ldots,t_p) \in T_i\}$$

and take $t^i = (v_1,\ldots,v_p)$.

Step 2. If $SA(t = t^i) = \varnothing$, then go to step 5.

If $SA(t = t^i) \neq \varnothing$, then choose $x^i \in SO(t = t^i)$ and put $\alpha = (f_1(x^i),\ldots,f_p(x^i))$.

Step 3. Increase j with 1, put $y^j = x^i$ and $k := 1$.

Step 4. If $\alpha_k > u_k$, then put $T_{h+1} := T_k(t = \alpha)$, increase h with 1 and go to step 5. If $\alpha_k = u_k$, then go to step 5.

Step 5. If $k = p$, then go to step 6. If $k \neq p$, then put $k := k+1$ and return to step 4.

Step 6. If $i < h$, then put $i := i + 1$ and return to step 1. If $i = h$, then the algorithm stops.

LEMMA 7. *If X is a bounded set and $j > 0$, then the points y_r for $r \in \{1, \ldots, j\}$ are min-efficient solutions for problem* v-min $(f; X)$.

Proof. Let $r \in \{1, \ldots, j\}$. From step 3 we get that there exists $i \in \{1, \ldots, h\}$ such that $y^r = x^i$. But x^i is an optimal solution for problem $P(t = t^i)$. Then, in view of corollary 2, x^i is a min-efficent solution for problem v-min $(f; X)$. It follows that y^r is also a min-efficient solution for problem v-min $(f; X)$.*

LEMMA 8. *If X is a bounded set and $j = 0$, then the problem* v-min $(f; X)$ *has not min-efficient solutions.*

Proof. If $SA(t = t^1) \neq \varnothing$, then, because X is a finite set, the problem $P(t = t^1)$ has also optimal solutions. In view of steps 2 and 3 it follows that $j > 0$. Therefore it is not possible that we have $SA(t = t^1) \neq \varnothing$. If $SA(t = t^1) = \varnothing$, then, in view of step 2 we go to step 5. Because $h = i = 1$, the algorithm is stopped. Hence, we have $j = 0$ if and only if the problem $P(t = t^1)$ has not feasible solution, that is the system

(24)
$$\begin{cases} f_1(x) \leq t_1^1 = b_1 \\ \ldots \\ f_p(x) \leq t_p^1 = b_p \\ x \in X \end{cases}$$

is inconsistent. From (12) we get that if $x \in X$, then

$$f_k(x) \leq b_k,\ \text{for each}\ k \in \{1,\ldots,p\}.$$

That implies that the system (24) is inconsistent because $X = \varnothing$. Therefore, min-$EF(f; X) = \varnothing$.*

THEOREM 9. *If z is an efficient solution for* v-min $(f; X)$, *then there is $i \in \{1, \ldots, h\}$ such that $z \in [x^i]$.*

Proof. Because for each $i \in \{1, \ldots, h\}$ we have $t^i \in T_i \subseteq T$, we get that $SA(t = t^i) = \varnothing$ if and only if $X = \varnothing$. Hence, in the case $X \neq \varnothing$, the stop of the algorithm implies $\alpha = a$.

Let z be an efficient solution for v-min $(f; X)$. In this case, in view of lemma 6, we get that $j \neq 0$. In view of theorem 1 there is $\beta \in T_0$ such that z is optimal solution for $P(t = \beta)$ and $s(z) = |\beta|$. From lemma 6 it results that there is $r \in \{1, \ldots, h\}$ such that

$\beta \in TL_r$ and $\beta \notin TL_s$ for each $r \in \{1, \ldots, h\}$ with $s > r$. Let x^r be the optimal solution for the problem $P(t = t^r)$ taking in the step 2. From corollary 5, the set $U(f(x^r), t^r)$ has the property that x^r is also an optimal solution for problem $P(t = \tau)$ for all $\tau \in U(f(x^r), t^r)$. Because

$$T_r = U(T^r, x^r) \bigcup \left(\bigcup_{p=r+1}^{h} \{t \in T_p : t \in T_r\} \right)$$

and $\beta \notin T_s$ for each $s > r$, it follows that $\beta \in U(f(x^r), t^r)$.

Since z is an optimal solution for problem $P(t = \beta)$ and $s(z) = |\beta|$, we get that

$$f_j(z) = \beta_j, \text{ for each } j \in \{1, \ldots, p\}. \tag{25}$$

From $x^r \in SO(t = \tau)$ for some $\tau \in U(t^r, x^r)$, we have

$$f_j(z^r) \le \beta_j,\ j \in \{1, \ldots, p\} \tag{26}$$

and

$$s(x^r) = s(z) = |\beta|. \tag{27}$$

From (26) and (27) it results

$$f_j(z) = f_j(x^r), \text{ for each } j \in \{1, \ldots, p\}. \tag{28}$$

Hence $z \in [x^r]$.*

THEOREM 10. *If X is a bounded set, then the number of equivalence classes is finite.*

Proof. If X is a bounded set, then $SZ = X \cap \mathbf{I}^n$ is a finite set. Because for some efficient solutions x for v-min$(f; X)$ we have $x \in X$, it is evident that the number of equivalence classes is finite.*

NUMERICAL EXAMPLE

To illustrate the algorithm we consider the following vectorial optimization problem:

$$\left\{ \begin{array}{l} v - \min f(x_1, x_2) = (-2x_1 + 3x_2^2,\ 3x_1^2 - 2x_2) \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ (x_1, x_2) \in \mathbf{I}^2 \end{array} \right. \tag{29}$$

We have $T = [-2,3] \times [-2,3]$. The corresponding parametric programming problem is

$$\left\{ \begin{array}{l} \min s(x_1, x_2) = 3x_1^2 + 3x_2^2 - 2x_1 - 2x_2 \\ -2x_1 + 3x_2^2 \le t_1 \\ 3x_1^2 - 2x_2 \le t_2 \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \\ (x_1, x_2) \in \mathbf{I}^2 \end{array} \right. \tag{30}$$

with $t \in T$.

Step 0. Put $i = 1$, $h = 1$, $j = 0$, $T_1 = [-2,3] \times [-2,3]$.

Step 1. We have $u_1 = -2$, $u_2 = -2$, $v_1 = 3$, $v_2 = 3$. We take $t_1 = (3,3)$.

Step 2. An optimal solution for the problem $P(t = (3,3))$ is $x^1 = (0,0)$. We take $\alpha = (0,0)$.

Step 3. We take $j = 1$, $y^1 = (0,0)$ and $k = 1$.

Step 4. Because $\alpha_1 = 0 < -2 = u_1$, we put $T_2 = [-2,1] \times [0,3]$, increase h with 1 and go to step 5.

Step 5. Because $k = 1 < 2$, increase k with 1 and return to step 4.

Step 4. Because $\alpha_2 = 0 > -2$, we put $T_3 = [-2, 3] \times [-2,-1]$, increase h with 1 and go to step 5.

Step 5. Since $k = 2$, we go to step 6.

Step 6. Because $i \ne h$, we increase i with 1 and we go to step 1.

Step 1. We take $u_1 = -2$, $u_2 = 0$, $v_1 = -1$, $v_2 = 3$, and $t^2 = (-1,3)$.

Step 2. An optimal solution for the problem $P(t = t^2)$ is $x^2 = (1,0)$. We take $\alpha = (-2,3)$.

Step 3. We take $j = 2$, $y^2 = (1,0)$, $k = 1$.

Step 4. Because $\alpha_1 = -2 = u_1$, we go to step 5.

Step 5. Since $k = 1 < 2$, we take $k = 2$ and we go to step 4.

Step 4. Because $\alpha_2 = 3 > u_2$, we take $T_4 = [-2, -1] \times [0,2]$, $h = 4$ and we go to step 5.

Step 5. Since $k = 2$, we go to step 6.

Step 6. Because $i = 2 < 4 = h$, we increase i with 1 and return to step 1.

Step 1. We take $u_1 = -2$, $u_2 = -2$, $v_1 = 3$, $v_2 = -1$ and $t_3 = (3, -1)$.

Step 2. $x^3 = (0,1)$ is an optimal solution for problem $P(t = t^3)$. We take $\alpha = (3, -2)$.

Step 3. We take $j = 3$, $y^3 = (0,1)$ and $k = 1$.

Step 4. Because $\alpha_1 > u_1$, we put $T_5 = [-2, 2] \times [-2, -1]$ and $h = 5$.

Step 5. We increase k with 1 and return to step 4.

Step 4. We increase k with 1.

Step 5. Because $k = 2$, we go to step 6.

Step 5. Since $i = 3 < h = 5$, we put $i = 4$ and we return to step 1.

Step 1. We take $u_1 = -2$, $u_2 = 0$, $v_1 = 2$, $v_2 = 2$ and $t_4 = (-1, 2)$.

Step 2. Because $SA(t = t^4) = \varnothing$, we go to step 6.

Step 5. Since $i = 4 < h = 5$, we increase i with 1 and we return to step 1.

Step 1. We take $u_1 = -2$, $u_2 = -2$, $v_1 = 2$, $v_2 = -1$ and $t^5 = (2, -1)$.

Step 2. Since $SA(t = t^5) = \varnothing$, we go to step 5.

Step 5. Because $i = 5 = h$, the algorithm stops.

The equivalence classes of efficient solutions for vectorial problem (29) are: $[y^1] = [(0, 0)]$, $[y^2] = [(1,0)]$, $[y^3] = [(0,1)]$.

REFERENCES

1. Dyer J.S, Sarin R.K., *Multicriteria decision making*. Mathematical Programming for Operations Researches and Computer Scientists. Industrial engineering. vol VI, New York and Basel: Marcel Dekker, Inc., 123–148.
2. Lupşa L., Duca E., Duca D.I., *On the structure of the set of points dominated and nondominated in an optimization problem*. Revue d'Anal. num. et la théorie de l'approximation, **22** (1993), *2*, 193–199.
3. Lupşa L., Duca D.I., Duca E., *On the Balanced and Nonbalanced Vector Optimization Problems*. Revue d'Anal. num. et la théorie de l'approximation, **25** (1995) *1*.
4. Podinovskey V.V., Nogin V.D., *Pareto-Optimal Solution of Multicriteria Problems* (in Russian). Nauka, Moscow, 1982.
5. Salukvadze M.E. and Topchishvili A.L., *Insoluble Multicriteria Linear Programming Problems*. Journal of Optimization Theory and Applications, **61**(1989), *3*, 487–491.
6. Salukvadze M.E. and Topchishvili A.L., *Weakly-Efficient Solutions of Limiting Multicriteria Optimization Problems*. Journal of Optimization Theory and Applications, **77**(1993), *2*, 373–386.
7. Sawaragi Y., Nakayama H., Tanino T., *Theory of Multiobjective Optimization*. Academic Press, San Diego – New York – London – Toronto – Montreal – Tokyo, 1985.
8. Steuer R.E., *Multiple-Criteria Optimization: Theory, Computation, and Applications*, John Wiley and Sons, New York, 1986.

Dorel I. Duca
"Babeş–Bolyai" University of Cluj-Napoca
Faculty of Mathematics and Informatics
3400 Cluj-Napoca, Romania

Eugenia Duca
Technical University
Department of Mathematics
3400 Cluj-Napoca , Romania

Liana Lupşa
"Babeş–Bolyai" University of Cluj-Napoca
Faculty of Mathematics and Informatics
3400 Cluj-Napoca, Romania