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ON A NONLINEAR INTEGRAL INEQUALITY ARISING IN THE THEORY OF DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

The following integral inequality has played a very important role in the theory of differential equations.

THEOREM A. *Let u and f be real-valued nonnegative continuous functions defined for* $t \geq 0$. *If*

$$u^2(t) \leq c^2 + 2\int_0^t f(s)\,u(s)\,\mathrm{d}s,$$

for all $t \geq 0$, *where* $c \geq 0$ *is a constant, then*

$$u(t) \leq c + \int_0^t f(s)\,\mathrm{d}s,$$

for all $t \geq 0$.

As far as we know, this inequality was first considered by L. Ou-Iang [7] in 1957 while studying the boundedness of the solutions of certain second order differential equations. In1979 C. M. Dafermos [3] used the following variant of the above inequality to establish a different connection between stability and second law of thermodynamics.

THEOREM B. *Assume that the nonnegative functions* $u \in L^\infty[0,s]$ *and* $g \in L^1[0,s]$ *satisfy the condition*

$$y^2(t) \leq M^2 y^2(0) + \int_0^t \left[2\alpha y^2(x) + 2Ng(x)y(x)\right]\mathrm{d}x,\ t \in [0,s],$$

where α, M, N *are nonnegative constants. Then*

$$y(s) \le Me^{\alpha s} y(0) + Ne^{\alpha s} \int_0^s g(x)\,dx.$$

The importance of these inequalities lies in their successful utilizations to the situations for which the other available inequalities do not apply directly. In the past few years the inequality given in Theorem A is used to obtain global existence, uniqueness, stability, boundedness and other properties of the solutions for wide classes of nonlinear differential equations, see [1-7, 10]. In view of the important role played by this inequality in the theory of differential equations, it is natural to expect that some new generalizations and extensions of this inequality would be equally important in certain new applications.

Our main objective here is to establish the two independent variables generalization of the inequality given in Theorem A which can be used as a handy tool in the analysis of certain classes of partial differential equations. The corresponding inequality on the discrete analogue of the main result is also established. Finally we present some immediate applications to convey the importance of our results to the literature.

2. STATEMENT OF RESULTS

In this section we state our main results to be proved in this paper. In what follows we denote by R the set of real numbers, $R_+ = [0, \infty)$ and $N_0=\{0, 1, 2, ...\}$. For any function $z(x, y)$ defined for $x, y \in R_+$, we denote the partial derivatives $\frac{\partial}{\partial x} z(x,y)$, $\frac{\partial}{\partial y} z(x,y)$, $\frac{\partial^2}{\partial y \partial x} z(x,y)$ by $z_x(x, y)$, $z_y(x, y)$, $z_{xy}(x, y)$ respectively. For any function $z(m, n)$ defined for $m, n \in N_0$, we define the operators $\Delta_1 z(m, n) = z(m+1, n) - z(m, n)$, $\Delta_2 z(m, n) = z(m, n+1) - z(m, n)$, and $\Delta_2\Delta_1 z(m, n) = \Delta_2[\Delta_1 z(m, n)]$, (see, [9]). For all $m > n$, $m, n \in N_0$ and any function $p(n)$ defined for $n \in N_0$ we use the usual conventions

$$\sum_{s=m}^{n} p(s) = 0 \quad \text{and} \quad \prod_{s=m}^{n} p(s) = 1.$$

Our main result is given in the following theorem.

THEOREM 1. *Let* $u(x,y)$, $f(x,y)$, $g(x,y)$ *be real-valued nonnegative continuous functions defined for* $x, y \in R_+$ *and* c *be a nonegative real constant. Let* $L: R_+^3 \to R_+$ *be a contiuous function which satisfies the condition*

$$\text{(L)} \qquad 0 \le L(x, y, v) - L(x, y, w) \le k(x, y, w)(v - w),$$

for $x, y \in R_+$ *and* $v \ge w \ge 0$, *where* $k: R_+^3 \to R_+$ *is a continuous function. If*

$$\text{(2.1)} \quad u^2(x,y) \le c^2 + 2\int_0^x \int_0^y \left[f(s,t)\, u(s,t)\, L(s,t,u(s,t)) + g(s,t)\, u(s,t) \right] dt\, ds,$$

for $x, y \in R_+$, *then*

$$\text{(2.2)} \qquad u(x,y) \le p(x,y) + q(x,y) \exp\left(\int_0^x \int_0^y f(s,t)\, k(s,t,p(s,t))\, dt ds \right),$$

for $x, y \in R_+$, *where*

$$\text{(2.3)} \qquad p(x,y) = c + \int_0^x \int_0^y g(s,t)\, dt ds,$$

$$\text{(2.4)} \qquad q(x,y) = \int_0^x \int_0^y f(s,t)\, L(s,t,p(s,t))\, dt\, ds,$$

for $x, y \in R_+$.

An interesting and useful discrete analogue of Theorem 1 is embodied in the following theorem.

THEOREM 2. *Let* $u(m,n)$, $f(m,n)$, $g(m,n)$ *be real-valued nonnegative functions defined for* $m, n \in N_0$ *and* c *a nonnegative real constant. Let* $H: N_0^2 \times R_+ \to R_+$ *be a function which satisfies the condition*

$$\text{(H)} \qquad 0 \le H(m,n,v) - H(m,n,w) \le M(m,n,w)(v - w),$$

for $m, n \in N_0$ *and* $v \ge w \ge 0$, *where* M *is a real-valued nonnegative function defined for* $m, n \in N_0$, $w \ge 0$. *If*

$$\text{(2.5)} \quad u^2(m,n) \le c^2 + 2\sum_{s=0}^{m-1} \sum_{t=0}^{n-1} \left[f(s,t) u(s,t) H(s,t,u(s,t)) + g(s,t) u(s,t) \right],$$

for $m, n \in N_0$, *then*

$$\text{(2.6)} \qquad u(m,n) \le a(m,n) + b(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=0}^{n-1} f(s,t) M(s,t,a(s,t)) \right],$$

for $m, n \in \mathbf{N}_0$, *where*

(2.7) $$a(m,n) = c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} g(s,t),$$

(2.8) $$b(m,n) = \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)H(s,t,a(s,t)),$$

for $m, n \in \mathbf{N}_0$.

3. PROF OF THEOREM 1

We first assume that c is positive and define a function $z(x,y)$ by

(3.1) $$z(x,y) = c^2 + 2\int_0^x\int_0^y \left[f(s,t)u(s,t)L(s,t,u(s,t)) + g(s,t)u(s,t)\right]dtds,$$

From (3.1) and using the fact that $u(x,y) \le \sqrt{z(x,y)}$, it is easy to observe that

(3.2) $$z_{xy}(x,y) \le 2\sqrt{z(x,y)}\left[f(x,y)L\left(x,y,\sqrt{z(x,y)}\right) + g(x,y)\right].$$

From (3.2) and the facts that $\sqrt{z(x,y)} > 0$, $z_x(x,y) \ge 0$, $z_y(x,y) \ge 0$ for $x, y \in R_+$, we observe that

$$\frac{z_{xy}(x,y)}{\sqrt{z(x,y)}} \le 2\left[f(x,y)L\left(x,y,\sqrt{z(x,y)}\right) + g(x,y)\right] + \frac{z_x(x,y)z_y(x,y)}{2\left(\sqrt{z(x,y)}\right)^3},$$

i.e.

(3.3) $$\frac{\partial}{\partial y}\left(\frac{z_x(x,y)}{\sqrt{z(x,y)}}\right) \le 2\left[f(x,y)L\left(x,y,\sqrt{z(x,y)}\right) + g(x,y)\right].$$

By keeping x fixed in (3.3) and setting $y = t$ and integrating with respect to t from 0 to y and then keeping y fixed in the resulting inequality and setting $x = s$ and integrating with respect to s from 0 to x we obtain

(3.4) $$\sqrt{z(x,y)} \le p(x,y) + \int_0^x\int_0^y f(s,t)L\left(s,t,\sqrt{z(s,t)}\right)dtds.$$

Define

(3.5) $$v(x,y) = \int_0^x\int_0^y f(s,t)L\left(s,t,\sqrt{z(s,t)}\right)dtds.$$

Differentiating (3.5) and then using the fact that $\sqrt{z(x,y)} \le p(x,y) + v(x,y)$ and the hypothesis (L) we observe that

$$v_{xy}(x,y) = f(x,y)L\left(x,y,\sqrt{z(x,y)}\right) \le$$
$$\le f(x,y)L(x,y,p(x,y) + v(x,y)) =$$

(3.6) $$= f(x,y)\left[L(x,y,p(x,y) + v(x,y)) - L(x,y,p(x,y))\right] + f(x,y)L(x,y,p(x,y)) \le$$
$$\le f(x,y)k(x,y,p(x,y))v(x,y) + f(x,y)L(x,y,p(x,y)),$$

By keeping x fixed in (3.6) and seting $y = t$ and integrating with respect to t from 0 to y and then keeping y fixed in the resulting inequality and setting $x = s$ and integrating with respect to s from 0 to x we obtain

(3.7) $$v(x,y) \le q(x,y) + \int_0^x\int_0^y f(s,t)k(s,t,p(s,t))v(s,t)dtds.$$

From (3.7) we observe that

(3.8) $$v(x,y) \le q_\varepsilon(x,y) + \int_0^x\int_0^y f(s,t)k(s,t,p(s,t))v(s,t)\,dtds,$$

where $q_\varepsilon(x,y) = \varepsilon + q(x,y)$ in which $\varepsilon > 0$ is an arbitrary small constant. Since $q_\varepsilon(x,y)$ is positive and monotone nondecreasing for $x, y \in R_+$, from (3.8) we observe that

(3.9) $$\frac{v(x,y)}{q_\varepsilon(x,y)} \le 1 + \int_0^x\int_0^y f(s,t)k(s,t,p(s,t))\frac{v(s,t)}{q_\varepsilon(s,t)}dtds.$$

Inequality (3.9) implies the estimate (see, [8,p. 492])

(3.10) $$v(x,y) \le q_\varepsilon(x,y)\exp\left(\int_0^x\int_0^y f(s,t)k(s,t,p(s,t))dtds\right).$$

By letting $\varepsilon \to 0$ in (3.10) we have

$$v(x,y) \le q(x,y)\exp\left(\int_0^x\int_0^y f(s,t)k(s,t,p(s,t))dtds\right). \tag{3.11}$$

The desired inequality in (2.2) now follows by using (3.11) in (3.4) and the fact that $u(x,y) \le \sqrt{z(x,y)}$.

If c is nonnegative, we carry out the above procedure with $c+\varepsilon_0$ instead of c, where $\varepsilon_0 > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon_0 \to 0$ to obtain (2.2). The proof is complete.

4. PROOF OF THEOREM 2

We assume that c is positive and define a function $z(m,n)$ by

$$z(m,n) = c^2 + 2\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left[f(s,t)u(s,t)H(s,t,u(s,t)) + g(s,t)u(s,t)\right]. \tag{4.1}$$

From (4.1) and using the fact that $u(m,n) \le \sqrt{z(m,n)}$ we observe that

$$\Delta_2\Delta_1 z(m,n) \le 2\sqrt{z(m,n)}\left[f(m,n)H\left(m,n,\sqrt{z(m,n)}\right) + g(m,n)\right]. \tag{4.2}$$

By using the fact that $\sqrt{z(m,n)} > 0$, $\Delta_1 z(m,n) \ge 0$, $\sqrt{z(m+1,n)} \le \sqrt{z(m+1,n+1)}$, $\sqrt{z(m,n)} \le \sqrt{z(m+1,n)}$, for $m, n \in \mathbf{N}_0$, it is easy to observe that

$$\Delta_1\left(\sqrt{z(m,n)}\right) = \frac{\Delta_1 z(m,n)}{\sqrt{z(m+1,n)} + \sqrt{z(m,n)}},$$

$$\Delta_2\Delta_1\left(\sqrt{z(m,n)}\right) = \frac{\Delta_1 z(m,n+1)}{\sqrt{z(m+1,n+1)} + \sqrt{z(m,n+1)}} - \frac{\Delta_1 z(m,n)}{\sqrt{z(m+1,n)} + \sqrt{z(m,n)}} \le \tag{4.3}$$

$$\le \frac{\Delta_2\Delta_1 z(m,n)}{\sqrt{z(m+1,n)} + \sqrt{z(m,n)}} \le \frac{\Delta_2\Delta_1 z(m,n)}{\sqrt{z(m,n)}}.$$

By using (4.2) in (4.3) we have

$$\Delta_2\Delta_1\left(\sqrt{z(m,n)}\right) \le \left[f(m,n)H\left(m,n,\sqrt{z(m,n)}\right) + g(m,n)\right]. \tag{4.4}$$

Now keeping m fixed in (4.4), set $n = t$ and sum over $t = 0, 1, 2, \ldots, n-1$ and then keeping n fixed in the resulting inequality set $m = s$ and sum over $s = 0, 1, 2, \ldots, m-1$, to obtain the following

$$\sqrt{z(m,n)} \le a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)H\left(s,t,\sqrt{z(s,t)}\right). \tag{4.5}$$

Define

$$v(m,n) = \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)H\left(s,t,\sqrt{z(s,t)}\right). \tag{4.6}$$

From (4.6) and using the fact that $\sqrt{z(m,n)} \le a(m,n) + v(m,n)$ and the hypothesis (H) we observe that

$$\begin{aligned}\Delta_2\Delta_1 v(m,n) &= f(m,n)H\left(m,n,\sqrt{z(m,n)}\right) \le \\ &\le f(m,n)\left[H(m,n,a(m,n)+v(m,n)) - H(m,n,a(m,n))\right] + \\ &\quad + f(m,n)H(m,n,a(m,n)) \le \\ &\le f(m,n)M(m,n,a(m,n))v(m,n) + f(m,n)H(m,n,a(m,n)).\end{aligned} \tag{4.7}$$

Now keeping m fixed in (4.7), set $n = t$ and sum over $t = 0, 1, 2, \ldots, n-1$ and then keeping n fixed in the resulting inequality set $m = s$ and sum over $s = 0, 1, 2, \ldots, m-1$ to obtain the following

$$v(m,n) \le b(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)M(s,t,a(s,t))v(s,t). \tag{4.8}$$

From (4.8) we observed that

$$v(m,n) \le b_\varepsilon(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)M(s,t,a(s,t))v(s,t), \tag{4.9}$$

where $b_\varepsilon(m,n) = \varepsilon + b(m,n)$, in which $\varepsilon > 0$ is an arbitrary small constant. Since $b_\varepsilon(m,n)$ is positive and monotone nondecreasing for $m, n \in \mathbf{N}_0$, from (4.9) we observe that

$$\frac{v(m,n)}{b_\varepsilon(m,n)} \le 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)M(s,t,a(s,t))\frac{v(s,t)}{b_\varepsilon(s,t)}. \tag{4.10}$$

Define

$$w(m,n) = 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)M(s,t,a(s,t))\frac{v(s,t)}{b_\varepsilon(s,t)}. \tag{4.11}$$

Form (4.11) and using (4.10) we observe that

$$\Delta_2\Delta_1 w(m,n) \le f(m,n)M(m,n,a(m,n))w(m,n). \tag{4.12}$$

From the definition of $w(m,n)$ we observe that $w(m,n) \le w(m,n+1)$, for $m, n \in \mathbf{N}_0$. Using this fact in (4.12) we observe that

$$\frac{\Delta_1 w(m,n+1)}{w(m,n+1)} - \frac{\Delta_1 w(m,n)}{w(m,n)} \le f(m,n)M(m,n,a(m,n)). \tag{4.13}$$

Now keeping m fixed in (4.13) and substituting $n = t$ and then taking the sum over $t = 0, 1, 2, \ldots, n-1$ and using the fact that $\Delta_1 w(m,0) = 0$, we have

$$\frac{\Delta_1 w(m,n)}{w(m,n)} \le \sum_{t=0}^{n-1} f(m,t)M(m,t,a(m,t)). \tag{4.14}$$

From (4.14) we observe that

$$w(m+1,n) \le w(m,n)\left[1 + \sum_{t=0}^{n-1} f(m,t)M(m,t,a(m,t))\right]. \tag{4.15}$$

Now keeping n fixed in (4.15) and letting $m = s$ and substituting $s = 0, 1, 2, \ldots, m-1$ successively, we have

$$w(m,n) \le \prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1} f(s,t)M(s,t,a(s,t))\right]. \tag{4.16}$$

Using (4.16) in (4.10) we get

$$v(m,n) \le b_\varepsilon(m,n)\prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1} f(s,t)M(s,t,a(s,t))\right]. \tag{4.17}$$

The desired inequality in (2.6) now follows by using (4.17) in (4.5) and then letting $\varepsilon \to 0$ in the resulting inequality and using the fact that $u(m,n) \le \sqrt{z(m,n)}$.

The proof of the case when c is nonnegative can be completed as mentioned in the proof of Theorem 1 and hence the proof is complete.

5. SOME APPLICATIONS

In this section, we present some applications of our results to obtain bounds on the solutions of certain differential and sum-difference equations. These applications are given as examples.

Example 1. As a first application we obtain a bound on the solution of the following partial differential equation

$$\left(z(x,y)z_x(x,y)\right)_y = z(x,y)\left[F(x,y,z(x,y)) + g(x,y)\right], \tag{5.1}$$

with the given boundary conditions

$$z(x,0) = \Phi(x), \quad z(0,y) = \Psi(y), \tag{5.2}$$

for $x, y \in R_+$, where $g: R_+^2 \to R$, $F: R_+^2 \times R \to R$, ϕ, $\psi: R_+ \to R$ are continuous functions and $\Phi(0) = \Psi(0)$. It is easy to observe that problem (5.1)-(5.2) is equivalent to the integral equation

$$z^2(x,y) = \mathrm{d}(x,y) + 2\int_0^x\int_0^y z(s,t)\left[F(s,t,z(s,t)) + g(s,t)\right]dt\,ds, \tag{5.3}$$

where $\mathrm{d}(x,y) = \Phi^2(x) + \Psi^2(y) - \Phi^2(0)$. If $z(x,y)$ is a solution of (5.1)–(5.2), then clearly it is also a solution of the integral equation (5.3). We assume that

$$|d(x,y)| \le c^2, \tag{5.4}$$

$$|F(x,y,z(x,y))| \le f(x,y)L(x,y,|z(x,y)|), \tag{5.5}$$

where $f(x,y)$ is a nonnegative real-valued continuous function defined for $x, y \in R_+$, c is a nonnegative real constant and $L: R_+^2 \times R_+ \to R_+$ is a continuous function satisfying the condition (L) in Theorem 1. From (5.3), (5.4) and (5.5) we observe that

$$|z(x,y)|^2 \le c^2 + 2\int_0^x\int_0^y \left[f(s,t)|z(s,t)|L(s,t,|z(s,t)|) + |g(s,t)||z(s,t)|\right]dt\,ds.$$

Now an application of Theorem 1 yields

$$(5.6)\quad |z(x,y)| \le p_1(x,y) + q_1(x,y)\exp\left(\int_0^x\int_0^y f(s,t)k(s,t,p_1(s,t))\,dtds\right),$$

where

$$p_1(x,y) = c + \int_0^x\int_0^y |g(s,t)|\,dtds,$$

$$q_1(x,y) = \int_0^x\int_0^y f(s,t)L(s,t,p_1(s,t))\,dtds,$$

for $x, y \in R_+$. Inequality (5.6) gives us the bound on the solution $z(x,y)$ of (5.1)–(5.2) in terms of the known functions.

Example 2. As a second application, we shall obtain a bound on the solution of the following sum-difference equation

$$(5.7)\quad z^2(m,n) = h(m,n) + 2\sum_{s=0}^{m-1}\sum_{t=0}^{n-1} z(s,t)\left[F(s,t,z(s,t)) + g(s,t)\right],$$

where $h, g: \mathbf{N}_0^2 \to R$, $F: \mathbf{N}_0^2 \times R \to R$ are functions such that

$$(5.8)\quad |h(m,n)| \le c^2,$$

$$(5.9)\quad |F(m,n,z(m,n))| \le f(m,n)H(m,n,|z(m,n)|),$$

where $f(m,n)$ is a real-valued nonnegative function defined for $m, n \in \mathbf{N}_0$, c is a nonnegative real constant and $H: \mathbf{N}_0^2 \times R_+ \to R_+$ is a function satisfying the condition (H) in Theorem 2. From (5.7), (5.8) and (5.9) we observe that

$$|z(m,n)|^2 \le c^2 + 2\sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left[f(s,t)|z(s,t)|H(s,t,|z(s,t)|) + |g(s,t)||z(s,t)|\right].$$

Now an application of Theorem 2 yields

$$(5.10)\quad |z(m,n)| \le a_1(m,n) + b_1(m,n)\prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1} f(s,t)M(s,t,a_1(s,t))\right],$$

for $m, n \in \mathbf{N}_0$, where

$$a_1(m,n) = c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}|g(s,t)|,$$

$$b_1(m,n) = \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f(s,t)H(s,t,a_1(s,t)),$$

for $m, n \in \mathbf{N}_0$. Inequality (5.10) gives the bound on the solution $z(m,n)$ of equation (5.7) in terms of the known functions.

In concluding this paper, we note that the inequalities established in this paper can be extended very easily to more than two independent variables. The precise formulations of these inequalities is very close to that of given in Theorems 1 and 2 with suitable modifications and hence we do not discuss it here.

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