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COMMON FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS OF TYPE (A) SATISFYING A PHI RATIONAL INEQUALITY

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THEOREM 1. *Let C be a closed convex subset of a Banach space X. If T is a mapping of C into itself satisfying the inequality*

(1) $$\|Tx - Ty\| \le a\|x - y\| + b\|Tx - x\| + c\|Ty - y\|,$$

for all x, y in C, where $0 < a < 1$, $0 < c$, $c < b$ *and* $a+b+c = 1$, *then T has a fixed point in C.*

Mappings satisfying inequality (1) with $a = 1$ and $b = c = 0$ are called non-expansive and were considered by Kirk [10]. Mappings with $a = 0$ and $b = c = 1/2$ were considered by Wong [14].

More recently, Diviccaro et al. [1], Fisher et al. [2] and many others generalized Theorem 1 in many ways. The following theorem was proved in [1]:

THEOREM 2. *Let T and I be two weakly commuting mappings of a closed convex subset C of a Banach space X into itself satisfying the inequality*

$$\|Tx - Ty\|^p \le a\|Ix - Iy\|^p + (1-a)\max\left\{\|Tx - Ix\|^p, \|Ty - Iy\|^p\right\}$$

for all x, y in C, where $0 < a < 2^{-p-1}$ *and* $p \ge 1$. *If I is linear, non-expansive in C and such that I(C) contains T(C), then T and I have a unique common fixed point at which T is continuous.*

Recall that the mappings T and I are said to be weakly commuting if

$$\|TIx - ITx\| \le \|Ix - Tx\|$$

for all x in C [12].

The next two definitions were given by Jungck [5] (see also [5]–[9]) and Jungck, Murthy and Cho [6] respectively.

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DEFINITION 1. *Let A and S be mappings of a Banach space X into itself. Then $\{A,S\}$ is said to be a compatible pair if*

$$\lim_{n\to\infty} \|ASx_n - SAx_n\| = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some t in X.

DEFINITION 2. *Let A and S be mappings of a Banach space X into itself. Then $\{A, S\}$ is said to be a compatible pair of type (A)* if

$$\lim_{n\to\infty} \|ASx_n - S^2x_n\| = \lim_{n\to\infty} \|SAx_n - A^2x_n\| = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

for some t in X.

We now give some propositions to prove our main theorem. The proofs of these propositions follow along the lines of the proofs in [7].

PROPOSITION 1. *Let $\{A, S\}$ be a compatible pair of type (A). Then it is a compatible pair if A or S is continuous.*

PROPOSITION 2. *Let $\{A, S\}$ be a compatible pair. Then it is a compatible pair of type (A) if A and S are both continuous.*

The following proposition is a direct consequence of Proposition 1 and 2.

PROPOSITION 3. *Let A and S be continuous mappings. Then $\{A, S\}$ is a compatible pair of type (A) if and only if it is a compatible pair.*

Example 1. Let $X = [0, 2]$ with the Euclidean norm and let A and S be the mappings defined by

$$Ax = \begin{cases} 2-x, & x \in [0,1) \\ 2, & x \in [1,2], \end{cases}$$

$$Sx = \begin{cases} x, & x \in [0,1) \\ 2, & x \in [1,2]. \end{cases}$$

Then A and S are not continuous at $t = 1$. We assert that $\{A, S\}$ is a compatible pair of type (A) but not a compatible pair.

To see this, suppose that $\{x_n\}$ is a sequence in X and that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t.$$

From the definition of A and S, it follows that $t \in [1, 2]$. Since $A = S$ on $[1, 2]$, we may suppose that $x_n \to 1$ and $x_n \le 1$ for all n. Then

$$Ax_n = 2 - x_n - 1 \quad \text{from the right}.$$

$$Sx_n = x_n - 1 \quad \text{from the left}.$$

Thus, since $x_n < 1$ for all n.

$$ASx_n = Ax_n = 2 - x_n \to 1$$

and since $2 - x_n > 1$ for all n,

$$SAx_n = S(2 - x_n) = 2.$$

Consequently,

$$\|ASx_n - Sax_n\| \to 1,$$

but

$$\|ASx_n - S^2x_n\| = \|2 - x_n - x_n\| \to 0$$

and

$$\|SAx_n - A^2x_n\| = \|2 - A(2 - x_n)\| \to 0.$$

Thus $\{A, S\}$ is a compatible pair of type (A) but is not a compatible pair.

Example 2. Let X be the set of real numbers with the Euclidian norm and let A and S be the mappings defined by

$$Ax = \begin{cases} x^{-1}, & x \ne 0, \\ 1, & x = 0, \end{cases}$$

$$Sx = \begin{cases} x^{-3}, & x \ne 0, \\ 1, & x = 0. \end{cases}$$

Then A and S are not continuous at $t = 0$. We assert that $\{A, S\}$ is a compatible pair but not a compatible pair of type (A).

To see this, consider the sequence $\{x_n\}$, where $x_n = n^2$ for all n. Then

$$Ax_n = n^{-2} \to 0, \quad Sx_n = n^{-6} \to 0$$

and

$$\|ASx_n - SAx_n\| = \|n^6 - n^6\| = 0.$$

Thus $\{A, S\}$ is a compatible pair.

However,

$$\|ASx_n - S^2x_n\| = \|n^6 - n^{18}\| \to \infty, \quad \|SAx_n - A^2x_n\| = \|n^6 - n^2\| \to \infty .$$

We therefore conclude that if A and S are discontinuous, then compatible mappings are not necessarily compatible mappings of type (A).

We also need the following proposition to prove our main theorem:

PROPOSITION 4. *Let A and S be mapping of X into itself. If A and S are compatible mappings of type (A) and $At = St$ for some t in X, then*

$$ASt = A^2t = S^2t = SAt.$$

PROPOSITION 5. *Let A and S be mappings of X into itself and let $\{A, S\}$ be a compatible pair of mappings of type (A). Suppose that the sequences $\{Ax_n\}$ and $\{Sx_n\}$ converge to t for some t in X. Then*

(a) $\lim_{n\to\infty} ASx_n = St$,

if S is continuous at t and

(b) $ASt = SAt$ and $St = At$,

if both A and S are continuous at t .

We now suppose that A, B, S and T are mappings from a Banach space X into itself such that

(1) $$A(X) \subseteq T(X), \quad B(X) \subseteq S(X),$$

and satisfying a phi rational inequality

(2) $$\|Ax - By\|^{p-q} \le \phi\left(\frac{a\|Sx - Ty\|^p + (1-a)\max\{\|Sx - Ax\|^p, \|Ty - By\|^p\}}{\max\{(a\|Sx - Ty\|^q + (1-a)\|Sx - Ax\|^q), \|Ty - By\|^q\}}\right)$$

for all x, y in X, where $0 < a < 1$, $p \ge 1$, $q \ge 0$, $p - q \ge 1$ and ϕ is a mapping of $[0, \infty)$ into itself such that ϕ is non-decreasing, upper semi-continuous and $\phi(t) < t$ for all $t > 0$.

Let x_0 be an arbitrary point in X. Then by (1) there exists a point x_1 in X such that $Ax_0 = Tx_1$ and then a point x_2 such that $Bx_1 = Sx_2$ and so on. We obtain a sequence (y_n) in X such that

(3) $$y_{2n} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1}$$

for $n = 0, 1, 2, \ldots$.

The following lemma was given by Singh and Meade [13]:

LEMMA 1. *Suppose that ϕ is a mapping of $[0, \infty)$ into itself which is nondecreasing upper semi-continuous and $\phi(t) < t$ for all $t > 0$. Then $\lim_{n\to\infty} \phi^n(t) = 0$.*

We now prove the following lemma:

LEMMA 2. *Let A, B, S and T be mappings of a Banach space X into itself satisfying conditions (1) and (3). Then the sequence $\{y_n\}$ as defined by (3) is a Cauchy sequence.*

Proof. Using (2) and (3) we have

$$\|y_{2n} - y_{2n+1}\|^{p-q} = \|Ax_{2n} - Bx_{2n+1}\|^{p-q} \le$$

$$\le \phi\left(\frac{a\|y_{2n-1} - y_{2n}\|^p + (1-a)\max\{\|y_{2n-1} - y_{2n}\|^p, \|y_{2n} - y_{2n+1}\|^p\}}{\max\{(a\|y_{2n-1} - y_{2n}\|^q + (1-a)\|y_{2n-1} - y_{2n}\|^q), \|y_{2n} - y_{2n+1}\|^q\}}\right).$$

If

$$\|y_{2n} - y_{2n+1}\| \ge \|y_{2n-1} - y_{2n}\|,$$

then

$$\|y_{2n} - y_{2n+1}\|^p \le \phi(\|y_{2n} - y_{2n+1}\|^p) < \|y_{2n} - y_{2n+1}\|^p,$$

a contradiction. Thus

$$\|y_{2n} - y_{2n+1}\|^p \le \phi(\|y_{2n-1} - y_{2n}\|^p).$$

Similarly we have

$$\|y_{2n+1} - y_{2n+2}\|^p \le \phi(\|y_{2n} - y_{2n+1}\|^p)$$

and it follows that

$$\|y_n - y_{n+1}\|^p \le \phi^n(\|y_0 - y_1\|^p)$$

for $n = 1, 2, \ldots$. It follows from Lemma 1 that

(4) $$\lim_{n\to\infty} \|y_n - y_{n+1}\| = 0.$$

In order to prove that $\{y_n\}$ is Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ and a sequence of even integers $n(k)$ defined inductively with $n(1) = 2$ and $n(k+1)$ is the smallest even integer greater than $n(k)$ such that

However,

$$\|ASx_n - S^2x_n\| = \|n^6 - n^{18}\| \to \infty, \quad \|SAx_n - A^2x_n\| = \|n^6 - n^2\| \to \infty.$$

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PROPOSITION 5. *Let A and S be mappings of X into itself and let $\{A, S\}$ be a compatible pair of mappings of type (A). Suppose that the sequences $\{Ax_n\}$ and $\{Sx_n\}$ converge to t for some t in X. Then*

(a) $\lim_{n\to\infty} ASx_n = St$,

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and satisfying a phi rational inequality

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for all x, y in X, where $0 < a < 1$, $p \ge 1$, $q \ge 0$, $p - q \ge 1$ and ϕ is a mapping of $[0, \infty)$ into itself such that ϕ is non-decreasing, upper semi-continuous and $\phi(t) < t$ for all $t > 0$.

Let x_0 be an arbitrary point in X. Then by (1) there exists a point x_1 in X such that $Ax_0 = Tx_1$ and then a point x_2 such that $Bx_1 = Sx_2$ and so on. We obtain a sequence (y_n) in X such that

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$$\le \phi\left(\frac{a\|y_{2n-1} - y_{2n}\|^p + (1-a)\max\{\|y_{2n-1} - y_{2n}\|^p, \|y_{2n} - y_{2n+1}\|^p\}}{\max\{(a\|y_{2n-1} - y_{2n}\|^q + (1-a)\|y_{2n-1} - y_{2n}\|^q), \|y_{2n} - y_{2n+1}\|^q\}}\right).$$

If

$$\|y_{2n} - y_{2n+1}\| \ge \|y_{2n-1} - y_{2n}\|,$$

then

$$\|y_{2n} - y_{2n+1}\|^p \le \phi(\|y_{2n} - y_{2n+1}\|^p) < \|y_{2n} - y_{2n+1}\|^p,$$

a contradiction. Thus

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Similarly we have

$$\|y_{2n+1} - y_{2n+2}\|^p \le \phi(\|y_{2n} - y_{2n+1}\|^p)$$

and it follows that

$$\|y_n - y_{n+1}\|^p \le \phi^n(\|y_0 - y_1\|^p)$$

for $n = 1, 2, \ldots$. It follows from Lemma 1 that

(4) $$\lim_{n\to\infty} \|y_n - y_{n+1}\| = 0.$$

In order to prove that $\{y_n\}$ is Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there is an $\varepsilon > 0$ and a sequence of even integers $n(k)$ defined inductively with $n(1) = 2$ and $n(k+1)$ is the smallest even integer greater than $n(k)$ such that

$$\|y_{n(k+1)} - y_{n(k)}\| > \varepsilon, \tag{5}$$

so that

$$\|y_{n(k+1)-2} - y_{n(k)}\| \le \varepsilon. \tag{6}$$

It follows that

$$\varepsilon < \|y_{n(k+1)} - y_{n(k)}\| \le$$

$$\le \|y_{n(k+1)} - y_{n(k+1)-1}\| + \|y_{n(k+1)-1} - y_{n(k+1)-2}\| + \|y_{n(k+1)-2} - y_{n(k)}\|$$

for $k = 1, 2, \dots,$ it follows that

$$\lim_{k\to\infty} \|y_{n(k+1)} - y_{n(k)}\| = \varepsilon. \tag{7}$$

By the triangular inequality, we have

$$\Big| \|y_{n(k+1)} - y_{n(k)}\| - \|y_{n(k)} - y_{n(k+1)-1}\| \Big| \le \|y_{n(k+1)} - y_{n(k+1)-1}\|$$

and

$$\Big| \|y_{n(k+1)-1} - y_{n(k)+1}\| - \|y_{n(k+1)} - y_{n(k)}\| \Big| \le \|y_{n(k+1)} - y_{n(k+1)-1}\| + \|y_{n(k)+1} - y_{n(k)}\|.$$

It follows from (6) and (7) that

$$\lim_{n\to\infty} \|y_{n(k)} - y_{n(k+1)-1}\| = \lim_{n\to\infty} \|y_{n(k+1)-1} - y_{n(k)+1}\| = \varepsilon. \tag{8}$$

Using (5), we have

$$\|y_{n(k+1)} - y_{n(k)}\| \le \|y_{n(k+1)} - y_{n(k)+1}\| + \|y_{n(k)+1} - y_{n(k)}\| =$$

$$= \|Ax_{n(k+1)} - Bx_{n(k)+1}\| + \|y_{n(k)+1} - y_{n(k)}\| \tag{9}$$

and using (4), we have

$$\|Ax_{n(k+1)} - Bx_{n(k)+1}\|^{p-q} \le \tag{10}$$

$$\le \phi\left(\frac{a\|y_{n(k+1)-1} - y_{n(k)}\|^p + (1-a)\max\left\{\|y_{n(k+1)-1} - y_{n(k+1)}\|^p, \|y_{n(k)} - y_{n(k)+1}\|^p\right\}}{\max\left\{a\|y_{n(k+1)-1} - y_{n(k)}\|^q + (1-a)\|y_{n(k+1)-1} - y_{n(k)}\|^q, \|y_{n(k)} - y_{n(k)+1}\|^q\right\}} \right).$$

Using (6), (7), (9), (10) and the upper semi-continuity of ϕ, it follows on letting n tend to infinity in (11) and (12) that

$$\left[\phi\left(\varepsilon^{p-q}\right)\right]^{1/p-q} < \varepsilon,$$

a contradiction. Therefore, $\{y_{2n}\}$ and so $\{y_n\}$ are Cauchy sequences in X.

We now prove our main theorem.

THEOREM 2. *Let A, B, S and T be mappings of a Banach space X into itself satisfying conditions* (1) *and* (2). *Suppose that one of the mappings A, B, S and T is continuous and that $\{A, S\}$ and $\{B, T\}$ are compatible pair of type* (*A*). *Then A, B, S and T have a unique common fixed point in X.*

Proof. Define the sequence $\{y_n\}$ as above. By Lemma 2, $\{y_n\}$ is a Cauchy sequence and has a limit u in X since X is a Banach space. Since $\{Ax_{2n}\}$ $\{Bx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ are subsequences of $\{y_n\}$, these subsequences converge to u.

Suppose S is continuous. Then

$$\lim_{n\to\infty} S^2x_{2n} = \lim_{n\to\infty} SAx_{2n} = Su.$$

Further, since $\{A, S\}$ is a compatible pair of type (A), we then have on using Proposition 5 that

$$\lim_{n\to\infty} ASx_{2n} = Su.$$

Using (2) we have

$$\|ASx_{2n} - Bx_{2n}\|^{p-q} \le$$

$$\le \phi\left(\frac{a\|S^2x_{2n} - Tx_{2n-1}\|^p + (1-a)\max\left\{\|S^2x_{2n} - ASx_{2n}\|^p, \|Tx_{2n-1} - Bx_{2n-1}\|^p\right\}}{\max\left\{a\|S^2x_{2n} - Tx_{2n-1}\|^q + (1-a)\|S^2x_{2n} - ASx_{2n}\|^q, \|Tx_{2n-1} - Bx_{2n-1}\|^q\right\}} \right).$$

Letting n tend to infinity we have

$$\|Su - u\|^{p-q} \le \phi\left(\|Su - u\|^{p-q} \le \|Su - u\|^{p-q},\right.$$

if $Su \ne u$, a contradiction. Thus $Su = u$.

Using (2) again we have

$$\|Au - Sx_{2n-1}\|^{p-q} \le \phi\left(\frac{a\|u - Tx_{2n-1}\|^p + (1-a)\max\left\{\|u - Au\|^p, \|Tx_{2n-1} - Bx_{2n-1}\|^p\right\}}{\max\left\{a\|u - Tx_{2n-1}\|^q + (1-a)\|u - Au\|^q, \|Tx_{2n-1} - Bx_{2n-1}\|^q\right\}} \right),$$

since $Su = u$. Letting n tend to infinity we have

$$\|Au - u\|^{p-q} \le \phi\left(\|u - Au\|^{p-q}, \le \|u - Au\|^{p-q},\right.$$

if $Au \ne u$, a contradiction. Thus $Au = u$.

Since $A(X) \subseteq T(X)$, there exists a point v in X such that $u = Au = Tv$. We claim that $Bv = u$, for if $Bv \ne u$, then using (2) we have

$$\|u - Bv\|^{p-q} = \|Au - Bv\|^{p-q} \le$$

$$\le \phi\left(\frac{a\|u - Tv\|^p + (1-a)\max\{\|u - Au\|^p, \|Tv - Bv\|^p\}}{\max\{a\|u - Tv\|^q + (1-a)\|u - Au\|^q, \|Tv - Bv\|^q\}}\right) =$$

$$= \phi\|u - Bv\|^{p-q} \le \|u - Bv\|^{p-q},$$

if $Bv \ne u$, a contradiction. Thus $Bv = u$.

Since $\{B, T\}$ is a compatible pair of type(A) on X and using proposition 4, we have $BTv = T^2v$ and so

$$Bu = BTv = T^2v = Tu.$$

Thus B and T have a coincidence point u.

We claim that u is in fact a common fixed point of B and T. Using (2) again, we have

$$\|u - Bu\|^{p-q} = \|Au - Bu\|^{p-q} \le$$

$$\le \phi\left(\frac{a\|Su - Tu\|p + (1-a)\max\{\|Au - Su\|^p, \|Bu - Tu\|^p\}}{\max\{a\|Su - Tu\|^q + (1-a)\|Au - Su\|^q, \|Bu - Tu\|^q\}}\right) =$$

$$= \phi\left(\|u - Tu\|^{p-q}\right) \le \|u - Tu\|^{p-q},$$

if $Tu \ne u$, a contradiction. Thus $Tu = u = Bu$ and we have proved that u is common fixed point of A, B, S and T.

The uniqueness of the common fixed point follows easily on using (2).

Remark. Theorem 2 generalizes the result of Murthy, Cho and Fisher [11].

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