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# THE PRESERVATION OF THE PROPERTY OF THE QUASICONVEXITY OF HIGHER ORDER BY BERNSTEIN'S OPERATORS

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(Braşov)

If $E \subset \mathbf{R}$, we denote by $\mathscr{F}(E)$ the set of the functions $f: E \to \mathbf{R}$ and for $n \in \mathbf{N}$, $n \geq 0$, we denote by $\mathscr{P}_n$ the set of the algebrical polynomials of degree not greater than $n$. If $f \in \mathscr{F}(E)$, $k \in \mathbb{N}$, $k \geq 0$ and $x_i \in E$, $(1 \leq i \leq k+1)$, $x_i \neq x_j$, $(i \neq j)$ we denote by $[x_1,\ldots,x_{k+1};f]$ the divided difference of order $k$ of the function $f$ on the points $x_i$.

We recall that a function $f \in \mathscr{F}(E)$ is said to be nonçoncave, (nonconvex respectively) of order $k$, $k \geq -1$ on $E$, if we have:

$$(1) \qquad [x_1,\ldots,x_{k+2};f] \geq 0 \ , \quad (\leq 0 \text{ respectively}),$$

for any system of distinct points $x_i \in E$, $(1 \leq i \leq k+2)$. In particular, the functions that are nonconcave (nonconvex) of order 0 coincide with the functions that are nondecreasing (nonincreasing) on $E$.

Bernstein's operators $B_n: \mathscr{F}([0,1]) \to \mathscr{P}_n$ are defined by:

$$(2) \quad B_n(f,x) = \sum_{k=0}^{n} f(k/n) p_{nk}(x), \quad \text{where} \quad p_{nk}(x) = \binom{n}{k} \cdot x^k (1-x)^{n-k}$$

$$f \in \mathscr{F}([0,1]), \ n \in \mathbb{N}, \ n \geq 1 \ \text{ and } \ x \in [0,1].$$

In [6] Tiberiu Popoviciu has proved that Bernstein's operators have the remarkable property that if $f \in \mathscr{F}([0,1])$ is nonconcave of order $k$ on $[0,1]$ then $B_n[f]$ is nonconcave of order $k$ on $[0,1]$, for any $n \geq 1$ and $k \geq -1$. In this paper we show that Bernstein's operators preserve also the property of the quasiconvexity of higher order of the functions. The notion of the quasiconvexity of higher order was introduced by Elena Popoviciu [4] in the following mode.

DEFINITION [4]. *The function* $f \in \mathcal{F}(E)$ *is quasiconvex of order k,* $k \geq 0$, *on E if the following inequality*

$$(3)\qquad [x_2,\ldots,x_{k+2};f] \leq \max\{[x_1,\ldots,x_{k+1};f],\ [x_3,\ldots,x_{k+3};f]\}$$

*holds for every system of points* $x_1 < \ldots < x_{k+3}$ *of* $E$.

Of course, if $E$ does not contain $k$+2 (respectively $k$+3) distinct points then, any functions $f \in \mathcal{F}(E)$ has automatically the above properties of convexity (respectively quasiconvexity) of order $k$.

The inequality in (3) is obviously equivalent with the following one

$$(4)\qquad \max\{[x_2,\ldots,x_{k+3};f],\ -[x_1,\ldots,x_{k+2};f]\} \geq 0.$$

In the case where $k = 0$ the inequality in (3) becomes:

$$(5)\qquad f(x_2) \leq \max\{f(x_1), f(x_3)\}.$$

In [7] Tiberiu Popovici has got the following simple characterization of the functions that satisfy (5). Here, the expression $E_1 < E_2$, where $E_1 \subset \mathbb{R}$, $E_2 \subset \mathbb{R}$, means that we have $x_1 < x_2$, for any $x_1 \in E_1$ and $x_2 \in E_2$..

THEOREM 1. [7] *A function* $f \in \mathcal{F}(E)$, *where E is an arbitrary subset of* $\mathbb{R}$, *satisfies the inequality in (5) for every points* $x_1 < x_2 < x_3$ *of E, if and only if there are two subset* $E_1$ *and* $E_2$ *such that* $E_1 \cup E_2 = E$, $E_1 < E_2$ *and f is nonincreasing on* $E_1$ *and nondecreasing on* $E_2$.

For our purpose we need to generalize this theorem. First we shall be concerned with the direct part of the equivalence. For this we use the following formula for the divided differences given in [5]: For every system of strictly ordered points $x_1 < \ldots < x_m$, $m \geq 2$, and every indices $1 = i_1 < \ldots < i_n = m$, $n \geq 1$, there are the real numbers $a_j \geq 0$, $(1 \leq j \leq m-n+1)$, $a_1 > 0$, $a_{m-n+1} > 0$ such that, for any function $f$ defined on the points $x_i$, $(1 \leq i \leq m)$ we have:

$$(6)\qquad [x_{i_1},\ldots,x_{i_n};f] = \sum_{j=1}^{m-n+1} a_j \cdot [x_j,\ldots,x_{j+n-1};f]$$

In what follows let the natural number $k \geq 0$ and the arbitrary subset $E \subset \mathbb{R}$ be fixed. Let us denote by $\Sigma$ the set of the all finite systems of points of $E$: $\sigma = \{x_1 < \ldots < x_m\}$, where $m \geq k+2$. For such a $\sigma$ and for $f \in \mathcal{F}(E)$ let us denote by $d_j = [x_j,\ldots,x_{j+k+1};f]$, $(1 \leq j \leq m-k-1)$, and let us denote by $\nu(\sigma)$ the greatest index $p, 1 \leq j \leq m-k-1$, such that $d_p < 0$. If there are not such indices we put $\nu(\sigma) = 0$. We have:



LEMMA. *If* $f \in \mathcal{F}(E)$ *is quasiconvex of order k,* $k \geq 0$, *on E then, the following inequalities hold:*

$$(7)\qquad \begin{aligned} &d_j \leq 0,\ (1 \leq j \leq \nu(\sigma)-1)\\ &d_{\nu(\sigma)} < 0\\ &d_j \geq 0,\ (\nu(\sigma)+1 \leq j \leq m-k-1). \end{aligned}$$

*Proof.* We have only to prove the first inequality in (7). Suppose that this one is not true and obtain a contradiction. Let $i$ be the greatest index $1 \leq i \leq \nu(\sigma)-1$ such that $d_i > 0$. We have $d_j \leq 0$ for any indices $i < j < \nu(\sigma)$, if there exist such indices. By applying relation (6) there are the numbers $a_j \geq 0$, $(i+1 \leq j \leq \nu(\sigma))$, $a_{i+1} > 0$, $a_{\nu(\sigma)} > 0$ such that:

$$[x_{i+1},\ldots,x_{i+k+1},x_{\nu(\sigma)+k+1};f] = \sum_{j=i+1}^{\nu(\sigma)} a_j \cdot d_j \leq a_{\nu(\sigma)} \cdot d_{\nu(\sigma)} < 0.$$

But this inequality together with $d_i > 0$ contradict the quasiconvexity of order $k$ of the function $f$ on the points $x_i < x_{i+1} < \ldots < x_{i+k+1} < x_{\nu(\sigma)+k+1}$ see (4). Lemma is proved. ■

We can prove moreover:

THEOREM 2. *If* $f \in \mathcal{F}(E)$ *is quasiconvex of order k,* $k \geq 0$, *where* $E \subset \mathbb{R}$ *is an arbitrary subset of* $\mathbb{R}$ *then, there are two subset* $E_1$ *and* $E_2$, *possibly one of them being the empty set, such that* $E_1 \cup E_2 = E$, $E_1 < E_2$ *and f is nonconvex of order k on* $E_1$ *and is nonconcave of order k on* $E_2$.

*Proof.* We suppose that $E$ contains at least $k$+3 points, since the contrary case is obvious. Let $a = \inf E$, $b = \sup E$, where the infimum may be $-\infty$ and the supremum may be $+\infty$.

If $\sigma = \{x_1 < \ldots < x_m\} \in \Sigma$ let us denote:

$$(8)\qquad \alpha(\sigma) = \begin{cases} x_{\nu(\sigma)}, & \text{if } \nu(\sigma) > 0\\ a & \text{if } \nu(\sigma) = 0 \end{cases}$$

Let us consider the set $A = \{\alpha(\sigma), \sigma \in \Sigma\}$, and let $\gamma = \sup A$, where $\gamma$ may be $\pm\infty$. We take the subset $E_1$ in the following mode:

$$(9)\quad \begin{array}{ll} \text{(i) } E_1 = (-\infty,\gamma)\cap E\,, & \text{if } \gamma \notin A \text{ or } \gamma = b \in A \\ \text{(ii) } E_1 = (-\infty,\gamma]\cap E\,, & \text{if } \gamma \in A \text{ or } \gamma < b, \end{array}$$

and then let $E_2 = E \setminus E_1$.

Let us prove that $f$ is nonconvex of order $k$ on $E_1$. We have only to consider the case where $E_1$ contains at least $k$+2 distinct points and let the points of $E_1 : x_1 < \ldots < x_{k+2}$. In each of the cases i) or ii) of (9) there is $\sigma \in \Sigma$ such that $x_{k+2} \le \alpha(\sigma) \le \gamma$. Let us put $\overline{\sigma} = \{x_1,\ldots,x_{k+2}\}\cup\sigma$ and let us represent $\overline{\sigma}$ by $\overline{\sigma} = \{y_1 < \ldots < y_m\}$. Denote $d_j = [y_j,\ldots,y_{j+k+1}; f]$, $(1 \le j \le m-k-1)$. Let $p$ be that index such that $y_p = \alpha(\sigma)$. It follows that $y_j \in \sigma$, for $p \le j \le m$. Therefore $d_p < 0$ and by (7) we have $d_j \le 0$, for $1 \le j \le p$. There are the indices $1 \le i_1 < \ldots < i_{k+2} \le p$ such that $x_s = y_{i_s}$, $(1 \le s \le k+2)$. By taking into account formula (6), there are the numbers $a_j \ge 0$, $(i_1 \le j \le i_{k+2} - k - 1)$, such that:

$$[x_1,\ldots,x_{k+2}; f] = \sum_{i_1 \le j \le i_{k+2}-k-1} a_j \cdot d_j \le 0.$$

Hence $f$ is nonconvex of order $k$ on $E_1$.

Let us prove now, that $f$ is nonconcave of order $k$ on $E_2$. We have also to prove only the case where $E_2$ contains at least $k$+2 distinct points. Let the points of $E_2 : x_1 < \ldots < x_{k+2}$. We have $[x_1,\ldots,x_{k+2}; f] \ge 0$. Indeed, contrarily we have $\alpha(\sigma) = x_1$, where $\sigma = \{x_1 < \ldots < x_{k+2}\}$ and hence $\gamma \le \alpha(\sigma)$. But this implies that $\gamma \in A$. By (9) and since $E_2$ contains $k+2 \ge 2$ distinct points it follows that $E_2 = (\gamma,\infty)\cap E$. Hence $x_1 > \gamma$, that means $\alpha(\sigma) > \gamma$. Contradiction. Then $f$ is nonconcave of order $k$ on $E_2$. ∎

*Remarks.* 1°. In [2] we obtain a variant of Theorem 2 in which the quasiconvexity of order $k$ is replaced by the more general notion of the quasiconvexity with respect to an interpolating set of functions. However in [2] it is required that $E$ be an interval.

2°. According to Theorem 1 the converse of the implication in Theorem 2 holds too, in the case $k = 0$. But in the general case $k \ge 1$ that is not true without supplementary conditions. For example, the function

$$(10)\qquad f(x) = \begin{cases} -x^3, & x \le 0 \\ x^3 + 5x\,, & x > 0\,, \end{cases}$$

is continous, it is nonconvex of order 2 on $(-\infty,0]$ and nonconcave of order 2 on $[0,\infty)$, but it is not quasiconvex of order 2, since $[-2,-1,0,1;f] = 1/6$ and $[-1,0,1,2;f] = -1/6$.



In a particular case we give the following reciprocal theorem:

THEOREM 3. *If $f: I \to \mathbb{R}$ is a polynomial function, where $I$ is an interval, and if there are the subintervals $I_1 < I_2$, $I_1 \cup I_2 = I$, such that $f$ is nonconvex or order $k \ge 0$ on $I_1$ and nonconcave of order $k$ on $I_2$, then $f$ is quasiconvex of order $k$ on $I$.*

*Proof.* If $I_1 = \varnothing$ or $I_2 = \varnothing$, then the theorem is obvious. Hence we consider $I_1 \ne \varnothing \ne I_2$ and let $c = \sup I_1 = \inf I_2$. From the hypothesis we have $f^{(k+1)}(x) \le 0$, $(x \in I_1)$ and $f^{(k+1)}(x) \ge 0, (x \in I_2)$. Hence $f^{(k+2)}(c) \ge 0$. There is a number $\delta > 0$ such that:

$$(11)\qquad f^{(k+2)}(x) \ge 0\,,\ \text{ if } |x-c| < \delta.$$

Indeed, the case where the degree of $f$ is not greater than $k$+2 or $f^{(k+2)}(c) > 0$ is obvious. In the opposite case we have $f^{(k+2)}(x) > 0$, for any $x, t_1 < x < t_2$, $x \ne c$, where by $t_1$ is denoted the greatest root of $f^{(k+2)}$ that is less than $c$ or $t_1 = -\infty$, if such root does not exist, and $t_2$ denotes the least root of $f^{(k+2)}$ that is greater than $c$, or $t_2 = +\infty$ if such root does not exist.

Consider now the points of $I: x_1 < \ldots < x_{k+3}$. In order to prove (3) it is enough to consider only the case where $x_1 < c < x_{k+3}$. We choose the points $y_1 < \ldots < y_m$ of $I$ with the following properties: 1° There are the indices $1 = i_1 < \ldots < i_{k+3} = m$ such that $y_{i_p} = x_p, (1 \le p \le k+3)$ and 2° $y_{j+1} - y_j < \delta/(k+2), (1 \le j \le m-1)$. We denote $c_j = [y_j,\ldots,y_{j+k}; f], (1 \le j \le m-k)$. Let $r \in \{2,\ldots,m\}$ be the least index such that $y_r \ge c$.

In what follows we shall use Cauchy's formula, i.e. for a function $g: J \to \mathbb{R}$, $J$ an interval, that admits the derivate of order $p$ on $J$, and for the points $t_1 < \ldots < t_{p+1}$ of $J$, there is $\xi$, $t_1 < \xi < t_{p+1}$ such that $[t_1,\ldots,t_{p+1}; g] = g^{(p)}(\xi)/p!$.

We prove the following relation:

$$(12)\qquad c_{j+1} \le \max\{c_j, c_{j+2}\}, \quad (1 \le j \le m-k-2).$$

If $1 \le j \le r-k-2$ it results $[y_j,\ldots,y_{j+k+1}; f] \le 0$ and hence $c_j \ge c_{j+1}$.

If $r-k-1 \le j \le \min\{r-1, m-k-2\}$ it results $|y_p - c| < \delta$, $(j \le p \le j+k+2)$, and hence, by (11) and by Cauchy's formula we infer $[y_j,\ldots,y_{j+k+2}; f] \ge 0$. Afterwards there results $[y_{j+1},\ldots,y_{j+k+2}; f] \ge [y_j,\ldots,y_{j+k+1}; f]$, and then:

$$c_{j+1} \le \frac{q}{p+q}\cdot c_j + \frac{q}{p+q}\cdot c_{j+2},$$

where $p = y_{j+k+1} - y_j > 0$ and $q = y_{j+k+2} - y_{j+1} > 0$. From the last inequality it obtains the inequality in (12).

Finally, if $r \le j \le m-k-2$ it results $[y_{j+1},\dots,y_{j+k+2}; f] \ge 0$ and hence $c_{j+1} \le c_{j+2}$. Thus in every cases (12) holds true.

Now take us an index $1 < i < m$ such that $y_i \notin \{x_1,\dots,x_{k+3}\}$ and consider the points $z_j$, $(1 \le j \le m-1)$ defined by: $z_1 = y_1,\dots,z_{i-1} = y_{i-1}, z_i = y_{i+1},\dots,z_{m-1} = y_m$. If we denote $d_j = [z_j,\dots,z_{j+k}; f]$, $(1 \le j \le m-k-1)$, by using (6) we can write $d_j = \lambda_j c_j + (1-\lambda_j)\cdot c_{j+1}$, where $\lambda_j \in [0,1]$, for any $1 \le j \le m-k-1$. Therefore from (12) and Theorem 1 it deduces

$$d_{j+1} \le \max\{d_j, d_{j+2}\},\quad (1 \le j \le m-k-3). \tag{13}$$

By repeating this procedure of elimination of the points that differ from the points $x_1, \dots, x_{k+3}$, we obtain after $m-k-3$ steps that (3) holds true. The theorem is proved. ■

Our main theorem is the following:

THEOREM 4. [1] *If* $f \in \mathcal{F}([0,1])$ *is quasiconvex of order k,* $k \ge 0$, *then for any* $n \ge 1$ *the polynomial function* $B_n[f]$ *is quasiconvex of order k.*

*Proof.* The following relation was proved in [6].

$$(B_n(f,x))^{(r)} = \frac{r!\,n!}{(n-r)!\,n^r}\sum_{i=0}^{n-r}\left[\frac{i}{n},\dots,\frac{i+r}{n}; f\right] p_{n-r,i}(x),\quad x \in [0,1] \tag{14}$$

We shall take $r = k+1$ in (14). Let us denote:

$$a_i = \left[\frac{i}{n},\dots,\frac{i+k+r}{n}; f\right]\frac{(k+1)!\,n!}{(n-k-1)!\,n^{k+1}}\binom{n-k-1}{i},\ (0 \le i \le n-k-1)$$

and take $x \in [0,1)$ and $y = \dfrac{x}{1-x}$. We have:

$$(B_n(f,x))^{(k+1)} = (1-x)^{n-k-1}\sum_{i=0}^{n-k-1} a_i \cdot y^i .$$



Because the theorem is obvious for $n < k+1$, we suppose that $n \ge k+1$. By virtue of Lemma, one of the following three cases holds true: 1° $a_i \le 0$ $(0 \le i \le n-k-1)$ or 2° $a_i \ge 0$, $(0 \le i \le n-k-1)$ or 3° there is $p$, $0 < p < n-k-1$ such that $a_i \le 0$, $(0 \le i \le p)$, $\min_{0\le i\le p} a_i < 0$ and $a_i \ge 0$, $(p+1 \le i \le n-k-1)$, $\max_{p+1\le i\le n-k-1} a_i > 0$.

In accordance with a well-known theorem of Descartes the polynomial $P(y) := \sum_{i=0}^{n-k-1} a_i y^i$ can have at most a positive root. Moreover, $P$ can have a positive root $y_0$ only in the case 3° and in this case we have $P(y) < 0$ $(0 < y < y_0)$, $P(y) > 0$, $(y_0 < y)$ since $\lim_{y\to\infty} P(y) = +\infty$. Hence it is true one of the following cases: 1° $(B_n(f,x))^{(k+1)} \le 0$, $(x \in [0,1])$ or 2° $(B_n(f,x))^{(k+1)} \ge 0$, $(x \in [0,1])$, or 3° there is $x_0$, $0 < x_0 < 1$ such that $(B_n(f,x))^{(k+1)} \ge 0$, $(x \in [0,x_0])$ and $(B_n(f,x))^{(k+1)} \ge 0$, $(x \in [x_0,1])$.

By taking into account Theorem 3 it results that $f$ is quasiconvex on order $k$ on $[0,1]$. ■

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