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AN ERROR ANALYSIS FOR THE STEFFENSEN METHOD UNDER GENERALIZED ZABREJKO-NGYEN-TYPE ASSUMPTIONS

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1. INTRODUCTION

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

(1) $$F(x) + G(x) = 0$$

where F, G are nonlinear operators defined on some convex subset D of a Banach space E_1 with values in a Banach space E_2. The operator F is assumed to be Fréchet-differentiable on D, whereas the differentiability of G is not assumed.

We will study the convergence of the Steffensen method

(2) $$x_{n+1} = x_n - \left[x_n, g(x_n); F\right]^{-1}\left(F(x_n) + G(x_n)\right) \quad (n \geq 0), \quad x_0 \in D$$

to a locally unique solution x^* of equation (1). Hear $g: D \subseteq E_1 \to E_2$, $\left[x, y; F\right] \in L(E_1, E_2)$ and satisfy the condition

(3) $$\left[x, y; F\right](y - x) = F(y) - F(x) \quad \text{for all} \quad x, y \in D.$$

Let $x_0 \in D$, we assume that $\left[x_0, g(x_0); F\right]^{-1}$ exists and

(4) $$\left\|\left[x_0, g(x_0); F\right]^{-1}\left(\left[x+h_1, y+h_2; F\right] - \left[x, y; F\right]\right)\right\| \leq A_1\left(t_1 + \|h_1\|, t_1\right) + A_2\left(t_2 + \|h_2\|, t_2\right),$$

(5) $$\left\|\left[x_0, g(x_0); F\right]^{-1}\left(G(v+h_3) - G(v)\right)\right\| \leq A_3\left(t_3 + \|h_3\|, t_3\right) = A\left(t_3 + \|h_3\|\right) - A(t_3),$$

and

(6) $$\left\|g(z + h_4) - g(z)\right\| \leq A_4\left(t_4 + \|h_4\|, t_4\right)$$

for all $x \in U(x_0,t_1) = \{x \in E_1 | \|x - x_0\| \leq t_1\}$, $y \in U(x_0,t_2)$, $v \in U(x_0,t_3)$, $z \in U(x_0,t_4)$, $\|h_1\| \leq R - t_1$, $\|h_2\| \leq R - t_2$, $\|h_3\| \leq R - t_3$, $\|h_4\| \leq R - t_4$ for some $R > 0$. The functions A_1, A_2, A_3, and A_4 are continuous in both variables, and such that if one of the variables is fixed then A_1, A_2, A_3, and A_4 are continuous in both variables, and such that if one of the variables is fixed, then A_1, A_2, A_3, and A_4 are increasing functions of the other on the interval $[0, R]$, with $A_1(0,0)=A_2(0,0)=A_3(0,0)=A_4(0,0)=0$. The function A is nondecreasing on $[0, R]$.

The case when $G = 0$, $A_1(t_1 + \|h_1\|, t_1) = c_1\|h_1\|$, $A_2(t_2 + \|h_2\|, t_2) = c_2\|h_2\|$, $A_4(t_4 + \|h_4\|, t_4) = c_4\|h_4\|$ for some positive constants c_1, c_2, and c_4 has already been studied in [5], [6] and the references there. Here, we provide an error analysis as well as error bounds on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^*\|$ for all $n \geq 0$. We also show how to choose the functions A_1, A_2, A_3 and A_4.

2. CONVERGENCE ANALYSIS

We will need to introduce the constants

(7) $$r_0 = 0, \quad r_1 = \|x_1 - x_0\|, \quad d = \|x_0 - g(x_0)\|,$$

(8) $$a = 1 - [A_1(R_1,0) + A_2(R,0) + A_2(d,0)], \quad R_1 \leq R,$$

(9) $$a_0^* = R_1 - [d + A_4(R_1,0)],$$

the sequences

(10) $$a_n = 1 - [A_1(r_n,0) + A_2(d + A_4(r_n,0),0) + A_2(d,0)],$$

(11) $$l_n = 1 - [A_1(\|x_n - x_0\|,0) + A_2(d + A_4(\|x_n - x_0\|,0),0) + A_2(d,0)],$$

(12) $$r_{n+1} = r_n + \frac{1}{a_n}\{[2A_1(r_{n-1},0) + A_2(r_n,0) + A_2(d + A_4(r_{n-1},0),0)](r_n - r_{n-1}) + A(r_n) - A(r_{n-1})\},$$

and the functions

(13) $$T(r) = r_1 + \frac{1}{b}\{[2A_1(r,0) + A_2(r,0) + A_2(d + A_4(r,0),0)]r + A(r)\}$$

where

(14) $$b = b(r) = 1 - [A_1(r,0) + A_2(d,0) + A_2(d + A_4(r,0),0)].$$

We will now state and prove the main result:

THEOREM 1. *Let F, G; $g: D \subseteq E_1 \to E_2$ be nonlinear operators satisfying conditions (4), (5) and (6). Assume:*

(i) *for $x_0 \in D$ the inverse of the linear operator $[x_0, y(x_0); F]$ exists;*

(ii) *there exists a minimum positive number R_1, such that*

(15) $$T(R_1) \leq R_1;$$

(iii) *the numbers R, R_1 are such that $R_1 \leq R$ and the constants a and a_0^* given by (8) and (9) respectively are $a > 0$, and $a_0^* \geq 0$;*

and

(iv) *the ball*

(16) $$U(x_0, R) \subseteq D$$

Then

(a) *the scalar sequence $\{r_n\}$ $(n \geq 0)$ generated by (12) is monotonically increasing and bounded above by its limit, which is number R_1;*

(b) *the sequence $\{x_n\}$ $(n \geq 0)$ generated by the Steffensen method (2) is well defined, remains in $U(x_0, R_1)$ for all $n \geq 0$, and converges to a solution x^* of equation $F(x) + G(x) = 0$, which is unique in $U(x_0, R)$ (if $G = 0$ on D).*

Moreover, the following estimates are true for all $n \geq 0$;

(17) $$\|x_n - x_{n-1}\| \leq r_n - r_{n-1},$$

(18) $$\|x_n - x^*\| \leq R_1 - r_{n-1}$$

(19) $$\|[x_0, g(x_0); F]^{-1}(F(x_n) + G(x_n))\| \leq \alpha_n \leq \beta_n, \text{ with}$$

$$\alpha_n = [2A_1(\|x_{n-1} - x_0\|,0) + A_2(\|x_n - x_0\|,0) + A_2(d + A_4(\|x_{n-1} - x_0\|,0),0)]\|x_n - x_{n-1}\| +$$

(20) $$+ A_3(\|x_{n-1} - x_0\| + \|x_n - x_{n-1}\|, \|x_{n-1} - x_0\|),$$

(21) $$\beta_n = [2A_1(r_{n-1},0) + A_2(r_n,0) + A_2(d + A_4(r_{n-1},0),0)](r_n - r_{n-1}) + A_3(r_n, r_{n-1}),$$

$$\|x_n - x^*\| \leq \frac{\overline{\alpha}_n}{c_n}, \text{ with } \overline{\alpha}_n = [2A_1(\|x_{n-1} - x_0\|,0) + A_2(\|x_n - x_0\|,0) +$$

(22) $$+ A_2(d + A_4(\|x_{n-1} - x_0\|,0),0)]\|x_n - x_{n-1}\| \quad (G = 0 \text{ on } D)$$

(23) $$c_n = 1 - [A_1(\|x^* - x_0\|,0) + A_2(\|x_n - x_0\|,0) + A_2(d,0)],$$

(24) $$\|x_{n+1} - x_n\| \leq \|x_n - x^*\| + \frac{p_n}{a_n},$$

and

$$p_n = \left[2A_1(\|x_n - x_0\|,0) + A_2(\|x^* - x_0\|,0) + A_2(d + A_4(\|x_n - x_0\|,0),0)\right]\|x^* - x_n\| +$$

(25) $$+A_3(\|x_n - x_0\| + \|x_n - x^*\|, \|x_n - x_0\|).$$

Proof. By (7), (12), (15) and the monotonicity of the functions A_1, A_2, A_3 and A_4, we deduce that the sequence $\{r_n\}$ $(n \geq 0)$ is monotonically increasing and nonnegative. Using (7), (12), (15) we can easily get that r_0, r_1 and $r_2 \leq R_1$. Let us assume that $r_k \leq R_1$ for $k = 0, 1, 2, \ldots, n$. Then by (12)

$$r_{n+1} \leq r_n + \frac{1}{a}\{[2A_1(r_{n-1},0) + A_2(r_n,0) + A_2(d + A_4(r_{n-1},0),0)](r_n - r_{n-1}) +$$
$$+A(r_n) - A(r_{n-1})\} \leq$$
$$\leq r_{n-1} + \frac{1}{a}\{[2A_1(R_1,0) + A_2(R_1,0) + A_2(d + A_4(R_1,0),0)](r_{n-1} - r_{n-2}) +$$
$$+A(r_{n-1}) - A(r_{n-2})\} +$$
$$+\frac{1}{a}\{[2A_1(R_1,0) + A_2(R_1,0) + A_2(d + A_4(R_1,0),0)](r_n - r_{n-1}) +$$
$$+A(r_n) - A(r_{n-1})\} =$$
$$= r_{n-1} + \frac{1}{a}\{[2A_1(R_1,0) + A_2(R_1,0) + A_2(d + A_4(R_1,0),0)](r_n - r_{n-2}) +$$
$$+A(r_n) - A(r_{n-2})\} \leq$$
$$\leq \cdots \leq r_1 + \frac{1}{a}\{[2A_1(R_1,0) + A_2(R_1,0) + A_2(d + A_4(R_1,0),0)]R_1 + A(R_1)\} =$$
$$= T(R_1) \leq R_1 \text{ by (15)}.$$

Hence, the scalar sequence $\{r_n\}$ $(n \geq 0)$ is bounded above by R_1. By (ii), R_1 is the minimum zero of equation $T(r) - r = 0$ in $(0, R_1]$, and from the above $R_1 = \lim_{n\to\infty} r_n$.

Using (7), we get $x_1 \in U(x_0, R_1)$, and (17) is true for $n = 0$. Let us assume that $x_k \in U(x_0, R_1)$, and (17) is true for $k = 0, 1, 2, \ldots, n$. We first show that $[x_k, g(x_k); F]$ is invertible. In fact, by the induction hypothesis,

(26) $$\|x_k - x_0\| \leq \sum_{j=1}^{k} \|x_j - x_{j-1}\| \leq \sum_{j=1}^{k} (r_j - r_{j-1}) = r_k - r_0 = r_k \leq R_1.$$

Also, we have

$$\|x_0 - g(x_k)\| \leq \|x_0 - g(x_0)\| + \|g(x_0) - g(x_k)\| \leq$$
$$\leq d + A_4(\|x_0 - x_k\|,0) \leq d + A_4(r_k,0) \leq$$
$$\leq d + A_4(R_1,0) \leq R_1,$$

since $a_0 > 0$. That is $g(x_k) \in U(x_0, R_1)$. Hence by (4), (5) and (6)

$$\left\|[x_0, g(x_0); F]^{-1}([x_k, g(x_k); F] - [x_0, g(x_0); F])\right\| \leq$$
$$\leq \left\|[x_0, g(x_0); F]^{-1}([x_k, g(x_k); F] - [x_0, x_0; F])\right\| +$$
$$+\left\|[x_0, g(x_0); F]^{-1}([x_0, x_0; F] - [x_0, g(x_0); F])\right\| \leq$$
$$\leq A_1(\|x_k - x_0\|,0) + A_2(\|x_0 - g(x_k)\|,0) + A_2(\|x_0 - g(x_0)\|,0) \leq$$
$$\leq A_1(r_k,0) + A_2(\|x_0 - g(x_0)\| + \|g(x_0) - g(x_k)\|,0) + A_2(d,0) \leq$$
$$\leq A_1(r_k,0) + A_2(d + A_4(r_k,0),0) + A_2(d,0) \leq$$

(27) $$\leq A_1(R_1,0) + A_2(d + A_4(R_1,0),0) + A_2(d,0) < 1$$

since $a > 0$ by (iii) and (8).

It now follows by the Banach lemma on invertible operators that

(28) $$\left\|[x_k, g(x_k); F]^{-1}[x_0, g(x_0); F]\right\| \leq \frac{1}{l_k} \leq \frac{1}{a_k} \leq \frac{1}{a}$$

where a, a_k, l_k are given by (8), (10) and (11) respectively for all $k \geq 0$

Using the estimates (2), (3), (4), (5), (6) and (28) we obtain in turn for all $k \geq 1$

$$\|x_{k+1} - x_k\| \leq \left\|[x_k, g(x_k); F]^{-1}[x_0, g(x_0); F]\right\| \left\|[x_0, g(x_0); F]^{-1}(F(x_k) + G(x_k))\right\| \leq$$
$$\leq \frac{1}{l_k}\left\|[x_0, g(x_0); F]^{-1}\{[F(x_k) - F(x_{k-1}) - [x_{k-1}, g(x_{k-1}); F](x_k - x_{k-1})] +\right.$$
$$\left. +(G(x_k) - G(x_{k-1}))\}\right\| \leq$$
$$\leq \frac{1}{a_k}\left\|[x_0, g(x_0); F]^{-1}\right\| \left\|\{[[x_{k-1}, x_k; F] - [x_0, x_0; F] + [x_0, x_0; F] -\right.$$

(29) $$\left. -[x_{k-1}, g(x_{k-1}); F]](x_k - x_{k-1}) + (G(x_k) - G(x_{k-1}))\}\right\| \leq \frac{\alpha_k}{a_k},$$

where α_k is given by (20). But

(30) $\alpha_k \le \left[2A_1(r_{n-1},0)+A_2(r_n,0)+A_2(d+A_4(r_{n-1},0),0)\right](r_n-r_{n-1})+A_3(r_n,r_{n-1})=\beta_k$

Hence by (29), (30), and (12)

$$\|x_{k+1}-x_k\| \le r_{k+1}-r_k\ ,$$

which shows (17) for all $n \ge 0$.

It now follows from (26), (28) and (17) that the Steffensen iteration $\{x_n\}$ $(n \ge 0)$ is Cauchy, well defined and remains in $U(x_0, R_1)$ for all $n \ge 0$. Hence, it converges to some x^* in such a way that (18) is satisfied. For $n=0$, (18) gives $x^* \in U(x_0, R_1)$. By taking the limit as $n \to \infty$ in (2) we obtain $F(x^*)+G(x^*)=0$, which shows that x^* is a solution of equation (1).

To show uniqueness, when $G=0$ on D, we assume that there exists another solution y^* of equation (1) in $U(x_0, R)$. Then as before we obtain

$$\left\|[x_0,g(x_0);F]^{-1}\left([x^*,y^*;F]-[x_0,g(x_0);F]\right)\right\| \le$$

$$\le \left\|[x_0,g(x_0);F]^{-1}\left([x^*,y^*;F]-[x_0,x_0;F]\right)\right\| +$$

$$+\left\|[x_0,g(x_0);F]^{-1}\left([x_0,g(x_0);F]-[x_0,x_0;F]\right)\right\| \le$$

$$\le A_1(\|x_0-x^*\|,0)+A_2(\|x_0-y^*\|,0)+A_2(d,0) \le$$

(31) $\le A_1(R_1,0)+A_2(R,0)+A_2(d,0)<1$ since $a_0^*>0$

by (ii) and (9).

It now follows from (31) that the linear operator $[x^*, y^*; F]$ is invertible. Using the approximation

$$F(y^*)-F(x^*)=[x^*,y^*;F](y^*-x^*)$$

provided that $G=0$ on D, we get $x^*=y^*$, which shows that x^* is the unique solution of equation (1) in $U(x_0, R)$.

Using the approximation

$$x_{n+1}-x_n = x^*-x_n+\left([x_n,g(x_n);F]^{-1}[x_0,g(x_0);F]\right)\cdot[x_0,g(x_0);F]^{-1}\cdot$$

$$\cdot\left\{\left[[x_n,x^*;F]-[x_0,x_0;F]+[x_0,x_0;F]-[x_n,g(x_n);F]\right](x^*-x_n)+(G(x^*)-G(x_n))\right\},$$

(3), (4), (5), (6), (26), (28) and the triangle inequality as above we get

$$\|x_{n+1}-x_n\| \le \|x^*-x_n\|+\frac{p_n}{a_n},$$

where p_n is given by (25), which shows (24) for all $n \ge 0$.

Moreover, using (4)–(6), we obtain

$$\left\|[x_0,g(x_0);F]^{-1}\left([x^*,x_0;F]-[x_0,g(x_0);F]\right)\right\| \le$$

$$\le \left\|[x_0,g(x_0);F]^{-1}\left([x^*,x_0;F]-[x_0,x_0;F]\right)\right\| +$$

$$+\left\|[x_0,g(x_0);F]^{-1}\left([x_0,g(x_0);F]-[x_0,x_0;F]\right)\right\| \le$$

$$\le A_1(\|x^*-x_0\|,0)+A_2(\|x_n-x_0\|,0)+A_2(d,0) \le$$

$$\le A_1(R_1,0)+A_2(R_1,0)+A_2(d,0)<1 \quad \text{since} \quad a>0.$$

Therefore, we get

(32) $$\left\|[x^*,x_n;F]^{-1}[x_0,g(x_0);F]\right\| \le \frac{1}{c_n},$$

where c_n is given by (23) for all $n \ge 0$.

Furthermore, using the approximation (if F=0 in D)

$$F(x_n)-F(x^*)=[x^*,x_n;F](x_n-x^*),$$

(20) and (32), we obtain

$$\|x_n-x^*\| \le \left\|[x^*,x_n;F]^{-1}[x_0,g(x_0);F]\right\| \left\|[x_0,g(x_0);F]^{-1}F(x_n)\right\| \le$$

$$\le \frac{\overline{\alpha}_n}{c_n} \quad \text{for all} \quad n \ge 0,$$

where

$$\overline{\alpha}_n=\left[2A_1(\|x_{n-1}-x_0\|,0)+A_2(\|x_n-x_0\|,0)+A_2(d+A_4(\|x_{n-1}-x_0\|,0),0)\right]\|x_n-x_{n-1}\|,$$

which shows (22) for all $n \ge 0$.

That completes the proof of the theorem.

Remarks (a) Let us denote the right hand side of (4) by $A_5(t_1+\|h_1\|,t_1,t_2+\|h_2\|,t_2)$. Then we can choose

(33) $$A_5(t_1+\|h_1\|,t_1,t_2+\|h_2\|,t_2)=\sup_{\substack{x \in U(x_0,t_1),\, y \in U(x_0,t_2)\\ \|h_1\| \le R-t_1,\, \|h_2\| \le R-t_2}} \left\|[x_0,g(x_0);F]^{-1}\left([x+h_1,y+h_2;F]-[x,y;F]\right)\right\|,$$

(34) $$A_3(t_3+\|h_3\|,t_3)=\sup_{\substack{v \in U(x_0,t_3)\\ \|h_3\| \le R-t_3}} \left\|[x_0,g(x_0);F]^{-1}(G(v+h_3)-G(v))\right\|$$

and

$$A_4\left(t_4+\|h_4\|,t_4\right)=\sup_{\substack{x\in U(x_0,t_4)\\ \|h_4\|\le R-t_4}}\left\|g(z+h_4)-g(z)\right\|. \tag{35}$$

Estimates (4), (5) and (6) will now follow from the above choices of functions A_1, A_2, A_3, A_4 and A_5.

One can refer to [3] for some applications of these ideas to the solution of integral operators.

(b) Let us assume that instead of (4), (5) and (6), the following conditions are true:

$$\left\|\left[x_0,g(x_0);F\right]^{-1}\left([x,y;F]-[z,z;F]\right)\right\|\le q_1(r)\|x-z\|+q_2(r)\|y-z\|, \tag{36}$$

$$\left\|\left[x_0,g(x_0);F\right]^{-1}\left(G(x)-G(y)\right)\right\|\le q_3(r)\|x-y\| \tag{37}$$

and

$$\left\|\left[x_0,g(x_0);F\right]^{-1}\left(g(x)-g(y)\right)\right\|\le q_4(r)\|x-y\| \tag{38}$$

for all $x, y, z\in U(x_0, r)$ and q_1, q_2, q_3 and q_4 nondecreasing functions on $[0,R]$. For example, we can choose

$$q_1(r)=q_2(r)=\sup_{x,y,z\in U(x_0,r)}\frac{\left\|\left[x_0,g(x_0);F\right]^{-1}\left([x,y;F]-[z,z;F]\right)\right\|}{\|x-z\|+\|y-z\|}, \tag{39}$$

$$q_3(r)=\sup_{x,y\in U(x_0,r)}\frac{\left\|\left[x_0,g(x_0);F\right]^{-1}\left(G(x)-G(y)\right)\right\|}{\|x-y\|}, \tag{40}$$

and

$$q_4(r)=\sup_{x,y\in U(x_0,r)}\frac{\left\|\left[x_0,g(x_0);F\right]^{-1}\left(g(x)-g(y)\right)\right\|}{\|x-y\|}. \tag{41}$$

PROPOSITION. *Let* $f: U^2(x_0, R)\to E_2$ *be a nonlinear operator satisfying*

$$\|f(x,y)-f(z,z)\|\le k_1(r)\|x-z\|+k_2(r)\|y-z\|, \tag{42}$$

for all $x, y, z\in U(x_0, r)$ $r\le R$, *and for some nondecreasing functions* k_1 *and* k_2 *on* $[0,R]$.
Then

$$\|f(x+h_1,y+h_2)-f(x,y)\|\le v_1(t_1+\|h_1\|)-v_1(t_1)+v_2(t_2+\|h_2\|)-v_2(t_2), \tag{43}$$

for all $x\in U(x_0,t_1), y\in U(x_0,t_2)$, $\|h_1\|\le R-t_1$, $\|h_2\|\le R-t_2$ *with* $v_1(r)=\int_0^r k_1(t)\mathrm{d}t$ *and* $v_2(r)=\int_0^r k_2(t)\mathrm{d}t$.

Proof. Let $x\in U(x_0,t_1), y\in U(x_0,t_2)$, $\|h_1\|\le R-t_1$ *and* $\|h_2\|\le R-t_2$. Using (42) for $m\in\mathrm{N}$, we obtain

$$\begin{aligned}\|f(x+h_1,y+h_2)-f(x,y)\|&\le\sum_{j=1}^{m}\Big\|f\left(x+m^{-1}jh_1,y+m^{-1}jh_2\right)-\\&\quad-f\left(x+m^{-1}(j-1)h_1,y+m^{-1}(j-1)h_2\right)\Big\|\le\\&\le\sum_{j=1}^{m}k_1\left(t_1+m^{-1}j\|h_1\|\right)m^{-1}\|h_1\|+\sum_{j=1}^{m}k_2\left(t_2+m^{-1}j\|h_2\|\right)m^{-1}\|h_2\|\le\\&\le v_1(t_1+\|h_1\|)-v_1(t_1)+v_2(t_2+\|h_2\|)-v_2(t_2)\ \text{ as }\ m\to\infty,\end{aligned}$$

by the monotonicity of k_1, k_2 and the definition of the Riemann integral.

That completes the proof of the proposition.

By (4)–(6), (36)–(38) and (42)–(43) we now deduce that another choice for the functions A_1, A_2, A_3 and A_4 can be

$$A_1\left(t_1+\|h_1\|,t_1\right)=\int_{t_1}^{t_1+\|h_1\|}q_1(t)\mathrm{d}t,$$

$$A_2\left(t_2+\|h_2\|,t_2\right)=\int_{t_2}^{t_2+\|h_2\|}q_2(t)\mathrm{d}t,$$

$$A_3\left(t_3+\|h_3\|,t_3\right)=\int_{t_3}^{t_3+\|h_3\|}q_3(t)\mathrm{d}t,$$

and

$$A_4\left(t_4+\|h_4\|,t_4\right)=\int_{t_4}^{t_4+\|h_4\|}q_4(t)\mathrm{d}t.$$

Remarks c) Let $G=0$, $q_1(r)=q_2(r)=e_1$ and $q_4(r)=e_2$, e_1, $e_2>0$. Then our results can be reduced to the ones obtained in [4]–[7]. Moreover, the choices of A_1, A_2, A_3 and A_4 given in (a) and (b) above show that $A_1(t_1+\|h_1\|,t_1)\le e_1\|h_1\|$, $A_2(t_2+\|h_2\|,t_2)\le e_2\|h_2\|$ and $A_4(t_4+\|h_4\|,t_4)\le e_4\|h_4\|$, which means that our estimates on the distances $\|x_{n+1}-x_n\|$ and $\|x_n-x^*\|$ can be sharper than the ones in [4]–[7] (and the references there) for all $n\ge 0$.

(d) Estimates (22) and (24) can be solved explicitly for $\|x_n - x^*\|$, when for example $q_1(r) = c_1$, $q_2(r) = c_2$, $q_3(r) = c_3$ and $q_4(r) = c_4$ on $[0,R]$ for some $c_1, c_2, c_3, c_4 > 0$. Estimate (22) will provide an upper bound on $\|x_n - x^*\|$, whereas (24) will provide a lower bound on $\|x_n - x^*\|$ for all $n \geq 0$.

(e) The uniqueness of the solution x^* of equation (1) in $U(x_0, R)$ was established only when $G = 0$ on D. We assume that $G \neq 0$ on D, and define the iterations

$$y_{n+1} = y_n - [x_0, g(x_0); F]^{-1}(F(y_n) + F(y_n)), \quad \text{for any} \quad y_0 \in U(x_0, R_1) \quad (n \geq 0)$$

$$z_{n+1} = z_n - [x_0, g(x_0); F]^{-1}(F(z_n) + F(z_n)), \quad z_0 = x_0 \quad (n \geq 0)$$

$$s_{n+1} = s_n + [A_1(R_1,0) + A_2(R_1,0) + A_2(d,0)](s_n - s_{n-1}) + A(s_n) - A(s_{n-1}) \quad (n \geq 1)$$

$$s_0 = 0, \; s_1 = \|y_1 - y_0\|$$

$$t_{n+1} = t_n + [A_1(R_1,0) + A_2(R_1,0) + A_2(d,0)](t_n - t_{n-1}) + A(t_n) - A(t_{n-1}) \quad (n \geq 1)$$

$$t_0 = R, \quad s_1 \leq t_1 < R,$$

$$\delta_n = (t_n - s_n) + [A_1(R_1,0) + A_2(R_1,0) + A_2(d,0)](s_{n-1} - t_{n-1}) + A(s_{n-1}) - A(t_{n-1}) + t_n - s_n \quad (n \geq 1),$$

and the function

$$T_1(r) = s_1 + [A_1(r,0) + A_2(r,0) + A_2(d,0)]r + A(r).$$

Moreover, we assume that in addition to the hypotheses of the above theorem, there exists a minimum positive number R_1^* such that

$$T_1(R_1^*) \leq R_1^*,$$

and

$$\delta_n \geq 0 \quad (n \geq 1).$$

Then as in the theorem above, we can show:

(i) The sequence $\{s_n\}$ $(n \geq 0)$ is monotonically increasing, whereas the sequence $\{t_n\}$ $(n \geq 0)$ is monotonically decreasing and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = R_1^* \leq R_1 \quad \text{and} \quad T_1(R_1) \leq R_1.$$

(ii) The sequence $\{z_n\}$ $(n \geq 0)$ is well defined, remains in $U(x_0, R_1^*)$ for all $n \geq 0$, and converges to a solution z^* of equation (1), which is unique in $U(x_0, R)$, with $z^* = x^*$.
Moreover, the following estimates are true:

$$\|z_n - z_{n-1}\| \leq s_n - s_{n-1} \quad (n \geq 1)$$

$$\|z_n - x^*\| \leq R_1^* - s_n \quad (n \geq 0)$$

and

$$\|z_n - y_n\| \leq t_n - s_n \quad (n \geq 0).$$

The condition on the sequence $\{s_n\}$ can be dropped if we define the sequences

$$\bar{s}_{n+1} = [A_1(R_1,0) + A_2(R_1,0) + A_2(d,0)]\bar{s}_n + A(\bar{s}_n), \quad \bar{s}_0 = 0$$

$$\bar{t}_{n+1} = [A_1(R_1,0) + A_2(R_1,0) + A_2(d,0)]\bar{t}_n + A(\bar{t}_n), \quad \bar{t}_0 = R$$

instead of the sequences $\{s_n\}$ and $\{t_n\}$ respectively. The conclusions (i) and (ii) will then also hold for the new sequences $\{\bar{s}_n\}$ and $\{\bar{t}_n\}$.

Moreover, the following estimates are true:

$$s_n - s_{n-1} \leq \bar{s}_n - \bar{s}_{n-1}$$

and

$$t_n - s_n \leq \bar{t}_n - \bar{s}_n \quad \text{for all} \quad n \geq 0.$$

(f) In the derivation of estimate (29), condition (3) was used to replace $F(x_k) - F(x_{k-1})$ by $[x_{k-1}, x_k; F] \cdot (x_k - x_{k-1})$. Instead, we can use the approximation

$$\int_0^1 F'(x_{k-1} + t(x_k - x_{k-1}))(x_k - x_{k-1}) = [x_{k-1} + t(x_k - x_{k-1}), x_{k-1} + t(x_k - x_{k-1}); F],$$

in which case a similar theorem can be proved if we just replace the sequence $\{r_n\}$ and the function T by the new ones given by

$$r_{n+1} = r_n + \frac{1}{a_n}\left\{\int_{r_{n-1}}^{r_n} (A_1(r_{n-1},0) + A_2(r_n,0))dt + (A_1(r_{n-1},0) + A_2(r_n,0))(r_n - r_{n-1}) + A(r_n) - A(r_{n-1})\right\} \quad (n \geq 1), \; r_0 = 0, \quad r_1 = \|x_1 - x_0\|$$

and

$$T(r) = r_1 + \frac{1}{b}\left\{\int_0^r (A_1(r,0) + A_2(r,0))dt + (A_1(r,0) + A_2(d + A_4(r,0),0))r + A(r)\right\}.$$

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