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AN INTEGRAL INEQUALITY

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1. Let f and g be integrable functions on $[a,b]$ such that $m_1 \le f \le M_1$, $m_2 \le g \le M_2$ where m_i, M_i are real constants. H. Grüss [4] has proved that the following inequality holds:

$$|T(f,g)| \le \frac{1}{4}(M_1 - m_1)(M_2 - m_2) \tag{1}$$

where $T(f,g) = A(fg) - A(f)A(g)$, $A(f) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

The constant 1/4 is the best possible.

P. L. Čebyšev [3] has proved that if f and g have bounded derivatives on $[a,b]$, then

$$|T(f,g)| \le \frac{(b-a)^2}{12} \sup_{[a,b]} |f'| \cdot \sup_{[a,b]} |g'| \tag{2}$$

Mean-value theorems for $T(f, g)$ and applications to Korovkin Approximation Theory are given in [2] where more general positive linear functionals A are considered; see also [1].

In this paper we present a generalization of Čebyšev's result. An analogous result is given in [6] (see also [8]).

2. Let $c > 0$ and $\lambda \ge 1$ be real numbers. Let p be an integrable function on $[a,b]$ such that $c \le p(x) \le \lambda c$ for all $x \in [a,b]$.

Denote $A(f;p) = \int_a^b f(x)p(x)\,\mathrm{d}x \,/ \int_a^b p(x)\,\mathrm{d}x$ and $T(f, g; p) = A(fg; p) - A(f; p)\,A(g;p)$.

Let $n \ge 1$ be an integer.

THEOREM. *Let f and g be real functions on $[a,b]$ such that $f^{(n-1)} \in \mathrm{Lip}_M\alpha$, $g^{(n-1)} \in \mathrm{Lip}_N\alpha$, where $M, N > 0$ and $0 < \alpha \le 1$ are given constants. Suppose that*

$$(3)\qquad f^{(k)}\left(\frac{a+b}{2}\right)=g^{(k)}\left(\frac{a+b}{2}\right)=0;\quad 1\le k\le n-1$$

(If $n=1$, one considers that $f^{(0)}\equiv f$ and condition (3) does not exist). *Then we have*

$$(4)\qquad |T(f,g;p)|\le\frac{\lambda MN}{\lambda+2\alpha+2n-2}\left(\frac{b-a}{2}\right)^{2\alpha+2n-2}\left(\prod_{1}^{n-1}\frac{1}{\alpha+i}\right)^2$$

Proof. First, we observe that

$$|T(f,f;p)|^2=\frac{1}{\left(\int_a^b p(t)\,\mathrm{d}t\right)^2}\left|\int_a^b p(t)\big(f(t)-A(f;p)\big)\left(f(t)-f\left(\frac{a+b}{2}\right)\right)\mathrm{d}t\right|^2\le$$

$$\le\frac{1}{\left(\int_a^b p(t)\,\mathrm{d}t\right)^2}\int_a^b p(t)|f(t)-A(f;p)|^2\,\mathrm{d}t\int_a^b p(t)\left|f(t)-f\left(\frac{a+b}{2}\right)\right|^2\mathrm{d}t=$$

$$=\frac{1}{\int_a^b p(t)\,\mathrm{d}t}T(f,f;p)\int_a^b p(t)\left|f(t)-f\left(\frac{a+b}{2}\right)\right|^2\mathrm{d}t$$

Therefore we have

$$(5)\qquad T(f,f;p)\le\frac{1}{\int_a^b p(t)\mathrm{d}t}\int_a^b p(t)\left|f(t)-f\left(\frac{a+b}{2}\right)\right|^2\mathrm{d}t$$

On the other hand,

$$(6)\qquad -M\left|t-\frac{a+b}{2}\right|^{\alpha}\le f^{(n-1)}(t)\le M\left|t-\frac{a+b}{2}\right|^{\alpha}$$

Using (3) and successive integration of (6) on $\left[x,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},x\right]$, we get

$$(7)\qquad \left|f(x)-f\left(\frac{a+b}{2}\right)\right|\le M\left(\prod_{1}^{n-1}\frac{1}{\alpha+i}\right)\left|x-\frac{a+b}{2}\right|^{\alpha+n-1}$$

Let φ be integrable on $[a,b]$, $q\le\varphi\le Q$. Denote $\mu=A(\varphi)$. J. Karamata [5] (see also [7]) has proved

$$(8)\qquad \frac{\lambda q(Q-\mu)+Q(\mu-q)}{\lambda(Q-\mu)+(\mu-q)}\le A(\varphi;p)\le\frac{q(Q-\mu)+\lambda Q(\mu-q)}{Q-\mu+\lambda(\mu-q)}$$

Using (5), (7) and the second inequality of (8) we get

$$(9)\qquad T(f,f;p)\le\frac{\lambda M^2}{\lambda+2\alpha+2n-2}\cdot\left(\frac{b-a}{2}\right)^{2\alpha+2n-2}\left(\prod_{1}^{n-1}\frac{1}{\alpha+i}\right)^2$$

Since it is known that

$$(10)\qquad |T(f,g;p)|^2\le T(f,f;p)T(g,g;p)$$

it remains only to combine (10) and (9) for f and g.

COROLLARY. *Suppose that* $f^{(n-1)}\in Lip_M 1$, $g^{(n-1)}\in Lip_N 1$ *and* $f^{(k)}\left(\frac{a+b}{2}\right)=g^{(k)}\left(\frac{a+b}{2}\right)=0$, $1\le k\le n-1$. *Then*

$$(11)\qquad |T(f,g)|\le\frac{MN(b-a)^{2n}}{2^{2n}(n!)^2(2n+1)}$$

The constant $1/2^{2n}(n!)^2(2n+1)$ *is the best possible.*

Indeed, if

$$f(x)=\begin{cases}-\dfrac{M}{n!}\left(\dfrac{a+b}{2}-x\right)^n, & x\in\left[a,\dfrac{a+b}{2}\right]\\[2ex] \dfrac{M}{n!}\left(x-\dfrac{a+b}{2}\right)^n, & x\in\left[\dfrac{a+b}{2},b\right]\end{cases}$$

$$g(x)=\begin{cases}-\dfrac{N}{n!}\left(\dfrac{a+b}{2}-x\right)^n, & x\in\left[a,\dfrac{a+b}{2}\right]\\[2ex] \dfrac{N}{n!}\left(x-\dfrac{a+b}{2}\right)^n, & x\in\left[\dfrac{a+b}{2},b\right]\end{cases}$$

then the equality in (11) is valid, i.e., the constant cannot be improved.

REFERENCES

1. Andrica, D., Badea C., *Grüss' inequality for positive linear functionals*, Periodica Math. Hung., 19 (1988), 155-167.
2. Andrica, D., Rasa, I., *Nets in* $M_+^1(X)$ *and mean-value theorems*, Sem. Math. Anal. Babeş-Bolyai Univ. Cluj-Napoca, Fac. Math. Preprint 7 (1985), 7-12.

3. Čebyšev, P. L., *Sur les expressions approximatives des intégrales definies par les autres prises entre les mêmes limites*. Proc. Math. Soc. Kharkov **2** (1882), 93-98 (Russian), translated in Oeuvres, V. **2**, 1907, 716-719.

4. Grüss, G., *Über das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\mathrm{d}x\int_a^b g(x)\mathrm{d}x$, Mat. Z. 39 (1935), 215-226.

5. Karamata, J., *O prvom stavu srednjih vrednosti odredenih integrala*. Glas srpske kraljevske akademije, CLIV Beograd (1935), 119-144.

6. Lupaş, A., *The best constant in an integral inequality*. Mathematica (Cluj) *15* (*38*) (1973), 219-222.

7. Lupaş, A., *On two inequalities of Karamata*, Univ, Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No 602 - No 633 (1978), 119-123.

8. Ostrowski, A. M., *An integral inequality*. Aeq. Mathematicae 4 (1970), 358 - 373.

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