REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome 25, N^{os} 1-2, 1996, pp. 207–216

NUMERICAL METHODS FOR QUASISTATIC FRICTIONAL CONTACT PROBLEM

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1. INTRODUCTION

The new models of friction and contact, in the last decade, are often based on friction laws which recognize the compliant microstructure of contact interface,and that were not only more physically realistic than classical theories, but which were also mathematically tractable.

The existence of a solution for quasistatic frictional contact problems with normal compliance law was proved by Anderson [5] using incremental formulations and, in presence of a time regularization, by Klarbring et al.[6] in a different manner. Rabier et al. [7] proved the existence and local (for sufficiently small friction coefficients) uniqueness of solutions for cases in which sliding contact occurs in a prescribed direction.

The present paper is a continuation of the analysis presented in [4], which consists in a numerical analysis of a quasistatic contact problem in linear elasticity with dry friction. The problem is intended to model the physical situation of two elastically deforming bodies that come into contact with friction obeying the normal compliance law.

First we give a classical and variational formulation of the continuous contact problem. After obtaining the continuous contact problem we derive the result and obtain an incremental formulation obtained by time discretization of the problem.

Then we consider a discrete variational formulation of the incremental problem using a perturbed Lagrangian functional.

Also, in the present paper is described a contact finite element in the three dimensional case, generalizing the two dimensional case considered by Ju and Taylor in [3] and by Wriggers and Simo in [8].

2. CLASSICAL AND VARIATIONAL FORMULATIONS OF THE PROBLEM

Let $\Omega^\alpha \subset R^N, \alpha = 1, 2, N = 2, 3,$ the domains occupied by two elastic bodies that come into contact with friction.

Let us denote by Γ^α the boundary of Ω^α and let $\Gamma_0^\alpha, \Gamma_1^\alpha, \Gamma_2^\alpha$ be open and disjoint parts of Γ^α so that $\Gamma^\alpha = \Gamma_0^\alpha \cup \Gamma_1^\alpha \cup \Gamma_2^\alpha$ with $\alpha = 1, 2$.

Assume that the bodies Ω^α are subjected to volume forces of density $f^\alpha = \left(f_1^\alpha, \dots, f_N^\alpha\right)$ on Ω^α, to surface tractions of density $t^\alpha = \left(t_1^\alpha, \dots, t_N^\alpha\right)$ on Γ_1^α and are hold fixed on Γ_0^α. We shall use the following notation for the normal and tangential components of the displacements and of the stress vector:

$$u_n^\alpha = u_i^\alpha n_i^1, u_{t_i}^\alpha = u_i^\alpha - u_n^\alpha n_i^1, \sigma_n^\alpha = \sigma_{ij}^\alpha n_i^\alpha n_j^1, \sigma_{t_i}^\alpha = \sigma_{ij}^\alpha n_j - \sigma_n^\alpha n_i^1,$$

where $i, j = 1,\dots,N$, $n^\alpha = \left(n_1^\alpha, \dots, n_N^\alpha\right)$ is the outward normal unit vector on Γ^α and the summation convention is used for i and j.

Find the field of displacements $u^\alpha = \left(u_1^\alpha, \dots, u_N^\alpha\right)$, for a time interval [0, T], defineds on Ω^α which satisfy the following equations and conditions:

– the equilibrium equation

(1) $$\sigma_{ij,j}^\alpha\left(u^\alpha\right) + f_i^\alpha = 0 \quad \text{in} \quad \Omega^\alpha \times (0, T)$$

– the constitutive equation

(2) $$\sigma_{ij}^\alpha = a_{ijkh}^\alpha \, \varepsilon_{kh}\left(u^\alpha\right) \text{ in } \Omega^\alpha$$

where $a_{ijkh}^\alpha = a_{jikh}^\alpha = a_{khij}^\alpha$ and $a_{ijkh}^\alpha \xi_{ij} \xi_{kh} \geq c\,|\xi|$, $\xi = \left(\xi_{ij}\right)$, and $\varepsilon_{kh}\left(u^\alpha\right) = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_h} + \frac{\partial u_h}{\partial X_k}\right)$, f_i = the components of body force per unit volume, assumed to be sufficiently smooth functions of $x = (x_1,\dots, x_N)$;

– the boundary conditions

(3) $$u_i^\alpha = 0 \text{ on } \Gamma_0^\alpha \times (0, T)$$
$$\sigma_{ij}^\alpha\left(u^\alpha\right) n_j = t_i^\alpha \text{ on } \Gamma_1^\alpha \times (0, T)$$

– the initial conditions

(4) $$u^\alpha(x, 0) = u_o^\alpha, \quad \dot{u}^\alpha(x, 0) = u_1^\alpha \quad \text{in} \quad \Omega^\alpha \quad \text{at} \quad t = 0$$

with u_o^α, u_1^α given smooth functions of x;

– the normal interface response

(5) $$\sigma_n\left(u^\alpha\right) = -c_n\left(u_n^1 - u_n^2 - g\right)_+^{m_n} \text{ on } \Gamma_2^\alpha \times (0, T)$$

with c_n and m_n material parameters (see [2]),

or $$\sigma_n = \frac{c_1 (1617646.152\,\sigma/m)^{c_2}}{5.589^{1+0.0711 c_2}} \exp\left[-\frac{1 + 0.0711 c_2}{(1.363\sigma)^2} d^2\right]$$

where ξ is the initial mean plan distance $d = \xi\, g$, c_1 and c_2 are mechanical constants expressing the nonlinear distribution of the surface hardness, σ and m are statistical parameters of the surface profile, representing respectively the RMS surface roughness and the mean asperity slope.

– the friction and contact conditions:

when $u_n^1 - u_n^2 \leq g \Rightarrow \sigma_T\left(u^\alpha\right) = 0$,

and when $u_n^1 - u_n^2 > g$,

(6) $$\left|\sigma_T\left(u^\alpha\right)\right| < c_T\left(u_n^1 - u_n^2 - g\right)_+^{m_T} \Rightarrow \dot{u}_t^1 - \dot{u}_t^2 = 0$$
$$\left|\sigma_T\left(u^\alpha\right)\right| = c_T\left(u_n^1 - u_n^2 - g\right)_+^{m_T} \Rightarrow \exists \lambda \geq 0 \quad \dot{u}_t^1 - \dot{u}_t^2 = -\lambda \sigma_T$$

Where c_n, m_n, c_T, m_T are material constants depending on interface properties, $b_+ = \max(0, b)$, $\dot{u}_t^\alpha$ is the tangential velocity of material particles on Γ_2^α and g is the initial gap between Γ_2^1 and Γ_2^2 measured along the outward normal direction to Γ_2^1.

The friction law (6) is a generalization of Coulomb's friction law, which is recovered if $m_n = m_T$. In such a case, $\mu = c_T / c_n$ is the usual coefficient of friction. Law (6) also describes a dependence of the friction coefficient on normal contact pressure.

The classical formulation of the quasistatic contact problem is as follows:

Problem P1. *Find the displacement field* u^α *which satisfies* (1) - (6) $\forall\ t \in [0, T]$, *where* f^α *and* t^α *are taken time dependent.*

It is known (see [9]) that a variational formulation of the problem P1 is the following inequality

Problem P2. *Find the function* $u = \left[u^1, u^2\right] : [0, T] \to V$ s. t.

(7) $$a(u(t), v - \dot{u}(t)) + j_n(u(t), v - \dot{u}(t)) +$$
$$+ j_t(u(t), v) - j_t(u(t), \dot{u}(t)) \geq L(v - \dot{u}(t)), \ \forall\ v \in V$$

with the initial condition:

(8) $$u(x, 0) = u_0, \quad \dot{u}(x, 0) = u_1$$

The following notation and definitions were also used:

(9) $$V=\left\{v^{\alpha}\in\left[H^{1}(\Omega^{\alpha})\right]^{N};\ v^{\alpha}=0 \text{ a.e. on } \Gamma_0^{\alpha}\right\}$$

the space of admissibile displacements

(10) $$a(u,v)=\sum_{\alpha=1,2}\int_{\Omega}a_{ijkh}^{\alpha}\,\varepsilon_{ij}^{\alpha}(u)\,\varepsilon_{kh}^{\alpha}(v)\,dx^{\alpha}$$

the virtual work produced by the action of the stress $\sigma_{ij}(u)$ on the strains ε_{ij}

(11) $$j_n(u,v)=\int_{\Gamma_2^1}c_n\left(u_n^1-u_n^2-g\right)_+^{m_n}v_n\,ds$$

– the virtual work produced by the normal contact pressure on the displacement v:

(12) $$j_t(u,v)=\int_{\Gamma_2^1}c_T\left(u_n^1-u_n^2-g\right)_+^{m_T}\left|v_T^1-v_T^2\right|ds$$

– the virtual power produced by the frictional force on the displacement v.

(13) $$L(v)=\sum_{\alpha=1,2}\int_{\Omega^{\alpha}}f_i^{\alpha}v_i^{\alpha}dx^{\alpha}+\sum_{\alpha=1,2}\int_{\Gamma_1^{\alpha}}t_i^{\alpha}\gamma(v_i^{\alpha})ds^{\alpha}$$

– virtual work produced by the external forces.

Here γ is the trace operator mapping $\left(H^1(\Omega)\right)^N$ onto $\left(H^{1/2}(\Omega)\right)^N$ which may be decomposed into normal component $\gamma_n(v)$ and tangential components $\gamma_T(v)$. For simplicity of notation, the latter are denoted as v_n and v_T, respectively. We also observe that the boundary integrals on Γ_2^{α} are well defined for $1\le m_n, m_T\le 3$ if $N=3$ and for $1\le m_n, m_T$ if $N=2$, because, for $v_i\in\left[H^1(\Omega)\right]^N$, $\gamma(v)\in\left[L^q(\Gamma_2^{\alpha})\right]^N$, with $1\le q\le 4$ for $N=3$, and with $1\le q$ for $N=2$. In the case $N=2$ $m_n\in[2, 3.33]$, these restrictions on m_n, m_T basically come from the embedding theorem (see [2]).

3. INCREMENTAL FORMULATION

Now we derive a time discretized approximation of the quasistatic problem P2. Let us consider a partition $(t^0, t^1, \ldots, t^n)$ of the time interval $[0,T]$; and an incremental formulation, obtained by using the backward finite diference approximation of the time derivative of u^{α}.

If we set $u^k=u(t^k)$, $\Delta u^k=u^{k+1}-u^k$, $\Delta t^k=t^{k+1}-t^k$, $L^k=L(t^k)$, $\Delta L^k=L^{k+1}-L^k$, $k=0,1,\ldots,n-1$ and if take $\dot u(t^{k+1})=\Delta u^k/\Delta t^k$ then we obtain, at each time t^k, the following quasi-variational inequality:

$\Delta u^k\in V$ and

(14) $$\begin{aligned}&a(\Delta u^k, v-\Delta u^k)+j_n(u^k+\Delta u^k, v-\Delta u^k)+jt(u^k+\Delta u^k, v)-\\&-jt(u^k+\Delta u^k,\Delta u^k)\ge \Delta L^k(v-\Delta u^k)-F(u^k, v-\Delta u^k)\end{aligned}\quad \forall\, v\in V$$

where $F(u^k, v-\Delta u^k)=a(u^k, v-\Delta u^k)-L^k(v-\Delta u^k)$

The time discretized approximation of the problem P2 is as follows.

Problem P3. *Find* $u^1,\ldots,u^{n-1}\in V$ *defined by* $u^0=u_0$, $u^{k+1}=u^k+\Delta u^k$, *where* $\Delta u^k\in V$ *is the solution of inequality* (14). *Thus for a given load history the quasistatic problem is approximated by a sequence of incremental problems* (14). *Although every problem* (14) *is a static one, it requires appropriate updating of the displacements, and the loads after each increment.*

4. FINITE ELEMENT APPROXIMATIONS OF THE CONTACT PROBLEM

We consider a discrete variational formulation of the incremental problem P3, using for the contact area a three nodes contact element for the two dimensional case (see [3], [8]).

In the three dimensional case a four node contact element consisting of three 'master' nodes and one 'slave' node, is employed.

In all numerical applications we derived a perturbed Lagrangian formulation for the case of frictional stick and for the case of frictional slide. For the case of frictional stick the perturbed Lagrangian functional for bodies in contact has the following form:

(15) $$\begin{aligned}\Lambda(u,\Sigma_n,\Sigma_t,\Sigma_\tau)&=\frac{1}{2}a(u,u)-L(u)+\Sigma_n^TG_n+\Sigma_t^TG_t+\Sigma_\tau^TG_\tau-\\&-\frac{1}{2\omega n}\Sigma_n^T\Sigma_n-\frac{1}{2\omega t}\Sigma_t^T\Sigma_t-\frac{1}{2\omega\tau}\Sigma_\tau^T\Sigma_\tau\end{aligned}$$

where u is the vector of nodal displacement, Σ_n, Σ_t, Σ_τ are the vectors of normal and tangential nodal contact forces, respectively, G_n, G_t, G_τ are the vectors of normal and tangential nodal gaps and ω_n, ω_t, ω_τ are the normal and tangential penalty parameters respectively.

The Newton-Raphson method was applied to the discrete variational formulations that can be derived from these perturbed Lagrangian functionals.

The normal vector on defined plane by the nodes l, 2 and 3 and respectively vectors, defined by directions of the node 1-2 and 1-3 will be:

$$n = \frac{(x_2 - x_1)(x_3 - x_1)}{\left|(x_2 - x_1)(x_3 - x_1)\right|}, \; t = \frac{x_2 - x_1}{|x_2 - x_1|}, \; \tau = \frac{x_3 - x_1}{|x_3 - x_1|} \tag{16}$$

where $x_1 = X_1 + u_1$, $x_2 = X_2 + u_2$, $x_3 = X_3 + u_3$ signify the current positions of master nodes; X_1, X_2, X_3 are reference coordinates and u_1, u_2, u_3 are current nodal displacements of points 1, 2 and 3.

In addition, we define the current 'surfaces coordinates' as following:

$$a_t = \frac{x_s - x_1}{|x_2 - x_1|} t, \quad a_\tau = \frac{x_s - x_1}{|x_3 - x_1|} \tau \tag{17}$$

in which $x_s = X_s + u_s$ denotes the current position of the slave node s. The normal and tangential gaps g_n, g_t, g_τ are defined as:

$$g_n = (x_s - x_1)n, \quad g_t = \left(a_t - a_t^0\right)|x_2 - x_1|, \quad g_\tau = \left(a_\tau - a_\tau^0\right)|x_3 - x_1|, \tag{18}$$

where a_t^0 and a_τ^0 are the old surface coordinates at the last time step known.

Note that the gap g depends on the slave node s as well as on the master nodes 1, 2 and 3. Thus, the variation of the gap is obtained according to

$$g = \frac{\mathrm{d}}{\mathrm{d}\alpha} g(x_s + \alpha\eta_s,\, x_1 + \alpha\eta_1, x_2 + \alpha\eta_2, x_3 + \alpha\eta_3) \tag{19}$$

where

$$\eta(\eta_1, \eta_2, \eta_3, \eta_s) \equiv \delta u(\delta u_1, \delta u_2, \delta u_3, \delta u_s) \tag{20}$$

With respect to finite element implementations, explicit matrix expressions for the Lagrangian multiplier formulation and the penalty formulation are derived as follows.

The discrete variational equation associated with (15) take the form:

$$\delta_u \Pi(u) + \Sigma_n^T \delta_u G_n + \Sigma_t^T \delta_u G_t + \Sigma_\tau^T \delta_u G_\tau = 0 \tag{21}$$

$$\delta\Sigma_n^T \left(-\frac{1}{\omega_n} \Sigma_n + G_n \right) = 0 \tag{22}$$

$$\delta\Sigma_t^T \left(-\frac{1}{\omega_t} \Sigma_t + G_t \right) = 0 \tag{23}$$

$$\delta\Sigma_\tau^T \left(-\frac{1}{\omega_\tau} \Sigma_\tau + G_\tau \right) = 0 \tag{24}$$

where $\Pi(u) = \frac{1}{2} a(u,u) - L(u)$ is the total potential energy of the bodies in contact, $\delta u G_n = \left(\delta_u g_n^1, \delta_u g_n^2, \ldots, \delta_u g_n^s\right)^T$, $\delta u G_t = \left(\delta_u g_t^1, \delta_u g_t^2, \ldots, \delta_u g_t^s\right)^T$, $\delta u G_\tau = \left(\delta_u g_\tau^1, \delta_u g_\tau^2, \ldots, \delta_u g_\tau^s\right)^T$, S = total number of slave nodes in contact $s = 1, 2, \ldots, S$, analogous for $\delta\Sigma_n$, $\delta\Sigma_t$, $\delta\Sigma_\tau$.

The variational of a typical nodal normal gap $g_n \in G_n$ take the form:

$$\delta g_n = \sum_{j=1}^{3} \frac{\partial g_n}{\partial u_s^j} \eta_s^j + \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial g_n}{\partial u_i^j} \eta_i^j$$

with the notation (20) and $c_n = \left(\frac{\partial g_n}{\partial u_s^1}, \frac{\partial g_n}{\partial u_s^2}, \frac{\partial g_n}{\partial u_s^3}, \frac{\partial g_n}{\partial u_1^1}, \ldots, \frac{\partial g_n}{\partial u_3^3} \right)$, $\eta = \left(\eta_s^1, \eta_s^2, \eta_s^3, \eta_1^1, \ldots, \eta_3^3\right)$, we obtain:

$$\delta g_n = \eta^T c_n$$

Similarly, the variation of a typical nodal tangential gap $g_t \in G_t$, $g_\tau \in G_\tau$ can be obtained according to

$$\delta g_t = \eta^T c_t, \; \delta g_\tau = \eta^T c_\tau$$

Moreover, the residual vector R_B and the tangent stiffness K_B associated , with the total potential energy of the contacting bodies simply read, result

$$\delta\Pi(u) = \eta^T R_B \text{ and } \delta R_B = \eta^T K_B$$

With, the convention: $\left(u^1, \ldots, u^{12}\right) = \left(u_s^1, u_s^2, u_s^3, u_1^1, \ldots, u_3^3\right)$ Eq.(21) become:

$$\eta^T \left[R_B + \sum_{s=1}^{S} \left(\sigma_n^s c_n^s + \sigma_t^s c_t^s + \sigma_\tau^s \sigma_\tau^s \right) \right] = 0 \tag{25}$$

and analogous for Eq.(22)-(24) where

$$\sigma_n \in \Sigma_n, \; \sigma_t \in \Sigma_t, \; \sigma_\tau \in \Sigma_\tau.$$

To applly Newton's iteration scheme, consistent linearization of Eq.(25) and those corresponding Eq.(23)-(24), at $(u, \Sigma_u, \Sigma_t, \Sigma_\tau)$ is performed and leads to

$$\left[\eta^T,\ \delta\Sigma_n^t, \delta\Sigma_t^T,\ \delta\Sigma_\tau^T\right]\left\{\begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2^T & B_2 & O & O \\ A_3^T & O & C_3 & O \\ A_4^T & O & O & D_4 \end{bmatrix}\begin{Bmatrix} \Delta u \\ \Delta\Sigma_n \\ \Delta\Sigma_t \\ \Delta\Sigma_\tau \end{Bmatrix} = -\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}\right\}$$

where

$$A_1 = K_B + \sum_{s=1}^{S}\left(k_n^s + k_t^s + k_\tau^s\right),\quad A_2 = \sum_{s=1}^{S} c_n^s,\quad A_3 = \sum_{s=1}^{S} c_t^s,\quad A_4 = \sum_{s=1}^{S} c_\tau^s,$$

$$B_2 = -\frac{1}{\omega_n}I,\quad C_3 = -\frac{1}{\omega_t}I,\quad D_4 = -\frac{1}{\omega\tau}I,\quad R_1 = R_B + \sum_{s=1}^{S}\left(\sigma_n^s c_n^s + \sigma_t^s c_t^s + \sigma_\tau^s c_\tau^s\right)$$

$$R_2 = -\frac{1}{\omega_n}\Sigma_n + G_n,\quad R_3 = -\frac{1}{\omega_t}\Sigma_t + G_t,\quad R_4 = -\frac{1}{\omega_\tau}\Sigma_\tau + G_\tau,$$

is the matrix zero, and

$$\left(k_n^s\right)ji = \frac{\partial c_n^{si}}{\partial u_j} = \frac{\partial^2 g_n^s}{\partial u_i \partial u_j},\quad \left(k_t^s\right)ji = \frac{\partial c_t^{si}}{\partial u_j} = \frac{\partial^2 g_t^s}{\partial u_i \partial u_j},\quad \left(k_\tau^s\right)ji = \frac{\partial c_\tau^{si}}{\partial u_j} = \frac{\partial^2 g_\tau^s}{\partial u_i \partial u_j}.$$

Finally after the discrete formulation within the framework FEM, a standard assembly procedure can be used to add the contact contributions of each contact element to the global tangent stiffness and residual matrix and thus we obtain:

$$K\,U = R \tag{26}$$

where $K = K_B + \sum_{s=1}^{S} K_C^s$, $R = -\left(R_B + \sum_{s=1}^{S} R_C^s\right)$, K_B, R_B are mechanical global tangent stiffness matrix and residual vector, K_C^s, R_C^s are mechanical contact contributions of contact nod s, $U = \left(\Delta u, \Delta\Sigma_n, \Delta\Sigma_t, \Delta\Sigma_\tau\right)^T$, S is the total number of the slave nodes. And for $\omega_n = \omega_t = \omega_\tau = \omega$, and $\sigma_n = \omega g_n$, $\sigma_t = \omega g_t$, $\sigma_\tau = \omega g_\tau$ result

$$K_C = \sum_{s=1}^{S} \omega\left(g_n^s k_n^s + g_t^s k_t^s + g_n^s k_\tau^s + c_n^{sT} c_n^s + c_t^{sT} c_t^s + c_\tau^{sT} c_\tau^s\right) \tag{27}$$

$$R_C = \sum_{s=1}^{S} \omega\left(g_n^{sT} c_n^s + g_t^{sT} c_t^s + g_\tau^{sT} c_\tau^s\right) \tag{28}$$

For the case of frictional slide the relation $|\Sigma_{\tan}| = \mu|\Sigma_n|$, where μ is the coefficient of friction and $\Sigma_{\tan}$ is the result force of the Σ_t and Σ_τ, forces in the tangent plane of the contact surface.

Note with β the angle between the sides $\overline{x_2 - x_1}$ and $\overline{x_3 - x_1}$; we obtain $\cos\beta = t\tau$ and $|\lambda_{\tan}| = \mu\sqrt{g_t^2 + g_\tau^2 + 2\varepsilon|g_t||g_\tau|\cos\beta}$ where $\varepsilon = \mathrm{sgn}\,(g_t\, g_\tau)$. As a direct consequence of Coulomb's friction law, it results $\mu\omega|g_n| = \omega r$, where $r = \sqrt{g_t^2 + g_\tau^2 + 2\varepsilon|g_t||g_\tau|\cos\beta}$ therefore $\lambda_t = \lambda\tan\frac{g_t}{r}\omega g_n = -\mu\,\mathrm{sgn}(g_t)\frac{g_t}{r}\omega g_n = -\mu\frac{|g_t|}{r}\omega g_n$, $\lambda_\tau = -\mu\frac{|g_\tau|}{r}\omega g_n$.

If we write $d_t = \frac{|g_t|}{r}$, $d_\tau = \frac{|g_\tau|}{r}$, $b_t = \frac{\partial d_t}{\partial u}$, $b_\tau = \frac{\partial d_\tau}{\partial u}$, from liniarized kinematics (i.e., by negleting nonlinear terms k_t and k_τ), we obtain: $K_C = \sum_{s=1}^{S}\left(SL_1^s + SL_2^S\right)$, with

$$SL_1^S = \omega\left(g_n^s k_n^s - \mu g_n d_t^s k_t^s - \mu g_n^s d_\tau^s k_\tau^s + c_n^{sT} c_n^s - \mu d_t^s c_n^{sT} c_t^s\right)$$

$$SL_2^S = \omega\left(-\mu d_t^s c_n^{sT} c_\tau^s - \mu g_n b_t^{sT} c_t^s - \mu g_n b_\tau^{sT} c_\tau^s\right), \text{ and}$$

$$R_C = \sum_{s=1}^{S}\omega\left(\mu g_n d_t^{sT} c_t^s + \mu g_n d_\tau^{sT} c_\tau^s - g_n^{sT} c_n^s\right)$$

CONCLUSIONS

We give numerical exemples in [4] and [11], the numerical solution is in good agreement with the Raous [10]. The computations have been carried out within the environment of the Finite Element Analysis Program (FEAP), see Zienkiewicz [12], using the contact finite element in 3D, presented in this paper.

The critical situations arise in transitions from sliding to adhesion because it is then that the most important changes in the solution occur. One simple remedy for these difficulties is to decrease the time step until two successive solutions are not too far apart.

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Received 14.11.1994

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