REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

Tome 25, N^{os} 1–2, 1996, pp. 235–245

OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE SYMMETRIC CONVEX PROGRAMMING

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1. INTRODUCTION

In this paper for a class of nonlinear multiobjective programming problems with symmetrically differentiable pseudo-monotonic objective functions we present optimality conditions of Weber type [24].

We establish also a sufficient optimality condition and a weak duality theorem for a max-min problem involving symmetric pseudo-convex objective functions and symmetric quasi-convex constraints. For this aim, we transpose some of the results of Weir and Mond [25] to this symmetric pseudo-convex max-min problem.

2. SYMMETRIC (GENERALIZED) CONVEX FUNCTIONS

In this section we will briefly summarize some basic definitions and properties of symmetrically differentiable functions. Beyond this, some results concerning the so-called symmetric pseudo and quasi-concave (convex) functions are considered. These classes of functions are generally nonlinear nonconcave and nondifferentiable. For further details we refer to Minch [12]. Various properties of the usual pseudo and quasi-concave (or pseudo and quasi-convex) differentiable functions have been presented by Mangasarian [10], Martos [11], among others. Interesting results was obtained in the pseudo-monotonic case, from which we refer a Dantzig-Wolfe decomposition method for quasi-monotonic programming [15], linearization procedures for pseudo-monotonic programming [1], [2], [13], [16], optimality and duality properties [9], [19], [20], [22]. Some applications of these classes of functions in the max-min programming are given in [17], [18].

Other extensions of the quasi-convex and pseudo-convex functions are given by R. Pini and S. Schaible [25], and S. Komlosi [7], by using the generalized monotonicity. Also, G. Giorgi, A. Guerraggio [5], G. Giorgi and E. Molho [7] and G. Giorgi and S. Mititelu [6], present several observations on generalized invex functions and their relationships with other classes of generalized convex functions including the quasi-convex and pseudo-convex functions.

In [23], we considered symmetric invex functions and we extended some of the Giorgi and Molho [7] results for this more general class of generalized convex functions.

First we recall that for a real function f of one real variable the symmetric derivative of f at x is defined as:

$$f^s(x) = \lim_{h\to 0} \left(f(x+h) - f(x-h)\right)/2h ,$$

provided this limit exists (see, e.g. [12]).

This idea was extended by Minch [12] to functions of several variables.

DEFINITION 2.1 (Minch [12]) Let x be an element in an open domain A in R^n and let f: $A \to R$. If there exists a linear operator $f^s(x)$ from R^n to R, called the symmetric derivative of f at x, such that for sufficiently small h in R^n

$$f(x+h) - f(x-h) = 2f^s(x)h + u(x,h)\|h\|,$$

where $u(x,h)$ is in R and $u(x, h) \to 0$ as $\| h \| \to 0$, then f is said to be symmetrically differentiable at x. If f has a symmetric derivative at each point x in A, then f is symmetrically differentiable on A.

The notions of symmetric gradient and symmetric derivative are analogous to those of ordinary gradient and directional derivative. For convenience we shall denote the symmetric gradient of a symmetrically differentiable function f at x by $f^s(x)$.

Minch [12] has shown that f is symmetrically differentiable at x, in A, then the symmetric gradient is of the form:

$$f^s(x) = (D^s f(x;\, e^1), \dots, D^s f(x;\, e^n)),$$

where $e^1, \dots, e^n$ is the natural basis for R^n and $D^s f(x;h)$ denote the symmetric derivative of f at x (in A) in the direction h (in R^n), that is :

$$D^s f(x;h) = \lim_{t\to 0} \frac{f(x+th) - f(x-th)}{2t} \tag{2.1}$$

Let $f: A \to R$ and $g: A \to R$ be symmetrically differentiable functions at $x \in A$. From Definition 2.1, it follows easily that:

i) $f+g$ is symmetrically differentiable at x and

$$(f+g)^s(x) = f^s(x) + g^s(x); \tag{2.2}$$

ii) if f and g are continuous at x and $g(x)$ is not equal with zero, then f/g is symmetrically differentiable at x and

$$(f/g)^s(x) = \frac{f^s(x)g(x) - f(x)g^s(x)}{g^2(x)} \tag{2.3}$$

The following definition generalizes the pseudo-convexity concept.

DEFINITION 2.2 (Minch [12]) Let B be a subset of A and x' a point in A. The function f is said to be symmetrically pseudo-convex or s-pseudo-convex at x' (with respect to B) if f is symmetrically differentiable at x' and for all x in B, $f^s(x')\,(x-x') \ge 0$ implies $f(x) \ge f(x')$.

The function f is s-pseudo-convex on A if it is s-pseudo-convex at each point of A. The function f is s-pseudo-concave if $-f$ is s-pseudo-convex.

Analogous to the ordinary notion of differentiable quasi-convexity can be considered the notion of symmetrically quasi-convex function.

DEFINITION 2.3 (Minch [12]) Let B be a subset of A and x' a point in A. The function f is said to be symmetrically quasi-convex or s-quasi-convex at x', (with respect to B) if f is symmetrically differentiable at x' and for all x in B, $f(x) \le f(x')$ implies that $f^s(x')\,(x-x') \le 0$.

The function f is s-quasi-convex on A if it is s-quasi-convex at each point of A. Also the function f is s-quasi-concave if $-f$ is s-quasi-convex.

Examples: 1. The function $f: R \to R$ defined by

$$f(x) = x, \text{ for } x < 1,$$
$$f(x) = 1, \text{ for } x \in [1,2],$$
$$f(x) = x - 1, \text{ for } x > 2,$$

is a s-quasi-convex function but it is not s-pseudo-convex.

2. The function $f_1 : R \to R$ defined by

$$f_1(x) = x, \text{ for } x < 1,$$
$$f_1(x) = 0.5\,(x + 1), \text{ for } x \in [1,3],$$
$$f_1(x) = x - 1, \text{ for } x > 3,$$

is both s-pseudo-convex and s-quasi-convex but it is not pseudo-convex.

3. The function $f_2 : R \in R$ defined by

$$f_2(x) = x, \text{ for } x < 1,$$
$$f_2(x) = 0, \text{ for } x = 1,$$
$$f_2(x) = 0.5\,(x + 1), \text{ for } x \in (1,3],$$
$$f_2(x) = x - 1, \text{ for } x > 3,$$

is s-pseudo-convex but it is not s-quasi-convex.

Next, it will be assumed that s-pseudo-convexity (or s-quasi-convexity) at a point is with respect to the definition domain of the function unless otherwise stated.

DEFINITION 2.4 (Minch [12]) Let B be a subset of A and let x' be a point in A. The function f is said to be s-pseudo-monotonic (s-quasi-monotonic) at x' (with respect to B) if is symmetrically differentiable at x' and both s-pseudoconvex and s-pseudo-concave (s-quasi-convex and s-quasi-concave).

Since, if f has an ordinary derivative at x, then f has a symmetric derivative at x and they are equal, the following property holds.

PROPOSITION 2.1 (i) *If f is pseudo-convex (pseudo-concave) then f is s-pseudo-convex (s-pseudo-concave).*

(ii) *If f is differentiable quasi-convex (quasi-concave) then f is s-quasi-convex (s-quasi-concave).*

(iii) *If f is pseudo-monotonic (differentiable quasi-monotonic) then f is s-pseudo-monotonic (s-quasi-monotonic).*

It is easy to see that the converse assertions of those stated in Proposition 2.1 are not true.

Next we give some useful properties of the symmetrically quasi and pseudo-convex functions.

PROPOSITION 2.2 (*Tigan* [22]) *Let f be a symmetrically differentiable and continuous function. If f is a s-quasi-convex function on a convex subset B of A, then f is quasi-convex on B.*

PROPOSITION 2.3 *If f is s-pseudo-convex and continuous on a convex subset B of A, then f is quasi-convex on B.*

Proof. Let x', x'' be two points in B such that $f(x') \le f(x'')$. Suppose there exists x^* in the interval (x', x'') such that $f(x^*) > f(x'')$. Then, since f is continuous, there exists

$$x^0 = t'x' + (1-t')x'', 0 < t' < 1,$$

such that

$$f(x^0) = \max\{f(x) | x \in [x', x'']\}.$$

Therefore, by s-pseudo-convexity of f, because $f(x') < f(x^0)$ it follows that

$$(x' - x^0) f^s(x^0) < 0,$$

so, we have

$$(1-t')(x'-x'') f^s(x^0) < 0. \tag{2.4}$$

Also, the inequality $f(x'') < f(x^0)$ implies that

$$(x'' - x^0) f^s(x^0) = -t'(x' - x'') f^s(x^0) < 0. \tag{2.5}$$

But (2.4) contradicts (2.5). Therefore f is quasi-convex on B. ■

CONJECTURE 2.3.1 *If f is s-pseudo-convex and continuous on a convex subset B of A, then f is s-quasi-convex on B.*

3. MULTIOBJECTIVE SYMMETRIC PSEUDO-MONOTONIC PROGRAMMING

Let f_k ($k \in I = \{1,2,...,p\}$) be arbitrary objective functions defined on the open subset D of R^n and let X be a nonempty subset of D. Then we consider the following multiobjective programming problem:

VP. Find

$$V\max(f_1(x), \ldots, f_p(x)), \tag{3.1}$$

subject to $x \in X$.

If f_k ($k \in I$) are s-pseudo-monotonic objective functions then *VP* is said to be a symmetric pseudo-monotonic multiobjective program. In (9.1), "*V*max" means that efficient points are regarded as optimal solutions to *VP*.

DEFINITION 3.1 *A* point $x^* \in X$ is said to be effcient solution for *VP* if and only if there does not exist another point $x' \in X$ such that :

$$f_k(x') \ge f_k(x^*), \text{ for all } k \in I \text{ and}$$
$$f_{k'}(x') > f_{k'}(x^*) \text{ for at least one } k' \in I.$$

The set of all efficient solutions to *VP* is denoted by $E(X)$.

DEFINITION 3.2 A point $x^* \in X$ is said to be weakly efficient solution for *VP* if and only if there does not exist another point $x' \in X$ such that :

$f_k(x') > f_k(x^*)$, for all $k \in I$.

Clearly, every efficient point for a multiobjective program *VP* is weakly efficient but not conversely

As it is done e.g. by Bitran and Magnanti [3] (see, also [24]) we will relate the problem *VP* under the assumption of symmetric differentiability to a linear approximation at a point $x^0 \in X$ of that problem, namely

$P(x^0)$. Find

$$V\max(f_1^s(x^0)x, \ldots, f_p^s(x^0)),$$

subject to $x \in X$.

The following Theorem 3.1 gives a fully symmetric relation between VP and $P(x^0)$. A similar result has shown to be true by Weber [24], who, however, restricted to the differentiable pseudo-monotonic case, and which generalized a result obtained by Tigan [21] for the linear fractional multiobjective programming.

THEOREM 3.1 *Let f_k ($k \in I$) be s-pseudo-monotonic and continuous functions. A point $x^* \in X$ is efficient for the symmetric pseudo-monotonic program VP if and only if x^* is effcient for $P(x^*)$.*

Proof. First, let $x^* \in X$ be efficient for VP. Then, there is no $x' \in X$ such that:

$f_k(x') \geq f_k(x^*)$, for all $k \in I$ and

$f_{k'}(x') > f_{k'}(x^*)$ for at least one $k' \in I$.

Let us suppose there is $x' \in X$, such that

(3.2) $$f_k^s(x') \geq f_k^s(x^*),\text{ for all } k \in I \text{ and}$$

(3.3) $$f_{k'}^s(x') > f_{k'}^s(x^*)\quad\text{for at least one } k' \in I.$$

But since f_k ($k \in I$) is s-pseudo-convex and hence it is s-quasi-convex, it results from (3.2) and (3.3) that

$f_k(x') \geq f_k(x^*)$, for all $k \in I$ and

$f_{k'}(x') > f_{k'}(x^*)$ for at least one $k' \in I$.

But this contradicts the fact that x^* is an efficient solution for $P(x^*)$.

Conversely, let $x' \in X$ be efficient for $P(x^*)$. Then there is no x' in X such that

(3.4) $$f_k^s(x') \geq f_k^s(x^*),\text{ for all } k \in I \text{ and}$$

(3.5) $$f_{k'}^s(x') > f_{k'}^s(x^*)\text{ for at least one } k' \in I.$$

By s-pseudo-concavity of f_k ($k \in I$), from (3.4) and (3.5), we conclude that there is no x' in X such that

$f_k(x') \geq f_k(x^*)$, for all $k \in I$ and

$f_{k'}(x') > f_{k'}(x^*)$ for at least one $k' \in I$,

i.e. x^* is efficient for VP. ∎

THEOREM 3.2 *Let f_k ($k \in I$) be s-pseudo-monotonic and continuous functions. A point $x^* \in X$ is weakly efficient for the symmetric pseudo-monotonic multiobjective program VP if and only if x^* is weakly efficient for $P(x^*)$.*

Proof. The proof of this theorem is similar to that of Theorem 3.1. ∎

4. OPTIMALITY CONDITIONS FOR SYMMETRIC PSEUDO-CONVEX MINIMAX PROBLEMS

In this section, we consider the following minimax problem:

MP. Find

$$\operatorname*{Min}_x \operatorname{Max}\{f_1(x),\dots,f_r(x)\}$$

subject to

$$g(x) \leq 0,$$

where $f_i : R^n \to R$, (i=1,2,...,r) and $g : R^n \to R^m$ are symmetric differentiable functions (see, e.g. [11]).

The principal purpose of this section is to establish a sufficient optimality condition for problem MP involving symmetric pseudo-convex objective functions and symmetric quasi-convex constraints. We also define a dual problem to MP and establish a weak duality theorem.To this effect, we transpose some of the results of Weir and Mond [25] to the symmetric pseudo-convex maximin problem MP.

If the general minimax problem MP has a finite optimal value, then it may be expressed as following equivalent problem:

EP. Find

min q

subject to

$$f(x) \leq q\, e$$

$$g(x) \leq 0,$$

where

$$f(x) = (f_1(x),\dots,f_r(x))',\quad g(x) = (g_1(x),\ \dots,\ g_m(x))',$$

$$e = (1,1,\dots,1) \in R^s \text{ and } q \in R.$$

The main result of this section is:

THEOREM 4.1 *Let f_i (i=1,2,...,r) be s-pseudo-convex and g s-quasi-convex. If there exist $x^* \in R^n$, $q^* \in R$, $v^* \in R^r$, $u^* \in R^m$, such that :*

(4.1) $$v^* f^s(x^*) + u^* g^s(x^*) = 0,$$

(4.2) $$v^*\big(f(x^*) - q^* e\big) = 0,$$

(4.3) $$u^* g(x^*) = 0,$$

(4.4) $$v^* \geq 0,\ v^* e = 1,\ u^* \geq 0,$$

where $f = (f_1, \ldots, f_r)$ *and* $e = (1, 1, \ldots, 1) \in R^r$, *then* x^* *is an optimal solution for problem MP.*

In this theorem f^s denotes the symmetric gradient of the function f.

This theorem generalizes a similar result obtained by Weir and Mond [25] in the case of pseudo-convex objective functions and quasi-convex constraints.

Proof. Suppose that (x^*, q^*) is not optimal solution for *EP*. Then there exists a feasible solution (x, q) for *EP* with $q < q^*$. Thus

$$f_i(x) \le q < q^*, \quad i=1, 2, \ldots, r$$

and hence

$$v^*_i f_i(x) \le v^*_i q^*, \quad i = 1, 2, \ldots, r$$

with at least one strict inequality, since by (4.4), v^* is not the null vector. Hence, by (4.2),

$$v^*_i f_i(x) \le v^*_i f_i(x^*), \quad i = 1, 2, \ldots, r$$

with at least one strict inequality.

Since f_i is assumed s-pseudo-convex, then, for each $i=1,2,\ldots,r$ and $v_i \ge 0$, $v_i f_i$ is s-pseudo-convex and

$$(x-x^*)^t (v^*_i f^s_i(x^*)) \le 0, \quad i = 1, 2, \ldots, r$$

with at least one strict inequality.

Hence

$$(x-x^*)^t (v^{*t} f^s(x^*)) < 0.$$

Then it follows from (4.1) that

$$(x - x^*)^t \left(u^{*t} g^s(x^*)\right) > 0. \tag{4.5}$$

From (4.3), since x is feasible for *EP*, it results

$$u^*_i g_i(x) - u^*_i g_i(x^*) \le 0, \quad i=1, 2, \ldots, m.$$

But symmetric quasi-convexity of g implies

$$(x-x^*)^t (u^*_i g^s_i(x^*)) \le 0, \quad i=1, 2, \ldots, m$$

and hence

$$(x-x^*)^t (u^{*t} g^s(x^*)) \le 0,$$

which contradicts (4.5).

Thus (x^*, q^*) is optimal for *EP* and x^* is optimal for *MP*. ■

In relation to *MP*, which is equivalent to *EP*, we consider the following dual program:

DMP. Find

max z

subject to

$$v_i\left(f_i(y) - z\right) \ge 0, \quad i = 1, 2, \ldots, r \tag{4.6}$$

$$v^t f^s(y) + u^t g^s(y) = 0 \tag{4.7}$$

$$u^t g(y) \ge 0 \tag{4.8}$$

$$v \ge 0, \; v^t e = 1, \; u \ge 0, \; z \in R \tag{4.10}$$

THEOREM 4.2 (*Weak Duality*) *Let* (q,x) *be a feasible solution for EP and let* (y, v, u, z) *be a feasible solution for DMP. If f is s-pseudo-convex and, for all feasible* (q, x, y, v, u, z) *the function* $u^t g$ *is s-quasi-convex then* $q \ge z$.

Proof: Suppose $q < z$. Then

$$f_i(x) < v, \quad i=1, 2, \ldots, r$$

and, therefore

$$v_i\,(f_i(x) - z) \le 0, \quad i=1, 2, \ldots, r$$

with at least one strict inequality, since by (4.10), v is not the null vector. From (4.6)

$$v_i f_i(x) \le v_i f_i(y), \quad i=1, 2, \ldots, r$$

with at least one strict inequality.

Since each f_i is s-pseudo-convex, it follows

$$(x-y)^t (v_i f^s_i(y)) \le 0, \quad i=1, 2, \ldots, r$$

with at least one strict inequality.

Therefore

$$(x-y)^t (v^t f^s(y)) < 0$$

and from (4.7)

$$(x-y)^t (u^t g^s(y)) > 0. \tag{4.11}$$

From feasibility of x for *EP* and from (4.8) and (4.9)

$$u^t g(x) - u^t g(y) \le 0$$

and since $u^t g$ is s-quasi-convex

$$(x-y)^t (u^t g^s(y)) \le 0$$

which contradicts (4.11). ■

5. CONCLUSIONS

For a class of multiobjective programming problems with symmetrically differentiable pseudo-monotonic objective functions we present optimality conditions of Weber type.

We generalize also some results of Weir and Mond [25], establishing a sufficient optimality condition and a weak duality theorem for a max-min problem involving symmetric pseudo-convex objective functions and symmetric quasi-convex constraints.

Finally, we note that some of Weber's results [24] concerning the linearization techniques for finding efficient solutions of pseudo-monotonic multiobjective programming with linear constraints can be extended to the symmetrically pseudo-monotonic case.

REFERENCES

1. Bector C.R., Jolly P.L., *Programming problems with pseudomonotonic objectives*, Optimization, **15** (1984), *2*, 217-219.
2. Bhatt S.L., *Linearization Technique for linear fractional and pseudomonotonic programs revisited*, Cahiers du CERO, **23** (1981), 53-56.
3. Bitran G.R., Magnanti T.L. , *The structure of admissible points with respect to cone dominance*, JOTA, **29** (1979), 573-614.
4. Dantzig G.B., *Linear programming and extensions*, Princeton University Press, Princeton, New-Jersey, 1963.
5. Giorgi G., Guerraggio A., *Various types of invex functions*, Dipartimento di Richerche Aziendali, Universita di Pavia, 1994.
6. Giorgi G., Mititelu S., *Invexity in nonsmooth programming*, Atii del Tredicesimo Convegno A.M.A.S.E.S., Verona, 1989, 509-520.
7. Giorgi G., Molho E., *Generalized invexity: Relationships with generalized convexity and applications to optimality and duality conditions*, in Proceedings of the Workshop held in Pisa, 1992, "Generalized Concavity for Economic Applications", ed. Piera Mazzoleni, 1992, 53-70.
8. Komlosi S., *Generalized Monotonicity of generalized Derivatives*, in Proceedings of the Workshop held in Pisa, 1992, "Generalized Concavity for Economic Applications", ed. Piera Mazzoleni, 1992, 1-6.
9. Kortanek K.O., Evans J.P., *Pseudo-concave Programming and Lagrange regularity*, Oper. Res., **15** (1967), 882-891.
10. Mangasarian O.L., *Nonlinear Programming*, New York et al., Mc Graw Hill, 1969.
11. Martos B., *Nonlinear Programming Theory and Methods*, Amsterdam-Oxford, North-Holland, 1975.
12. Minch R. A., *Applications of symmetric derivatives in mathematical programming*, Math. Prog., **1** (1971), 307-320.
13. Mond B., *Techniques for pseudo-monotonic programming*, LaTrobe University, Pure Math. Res. Paper **82**,*12*, Melbourne, 1982.
14. Pini R., Schaible S., *Some Invariance Properties of Generalized Monotone Maps*, in Proceedings of the Workshop held in Pisa, 1992, "Generalized Concavity for Economic Applications", ed. Piera Mazzoleni, 1992, 87-88.
15. Tigan S., *Sur une méthode de décomposition pour le problème de programmation monotone*, Rev. Analyse Numér. Théor. Approx., **12**, *1* (1983), 347-354.
16. Tigan S., *On the linearization technique for quasi-monotonic optimization problems*, Analyse Num. Théor. Approx., **12**, *1* (1983), 89-96.

17. Tigan S., *A quasimonotonic max-min programming problem with linked constraints*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca University, 1986, 279-284.
18. Tigan S., *On a quasimonotonic max-min problem*, Analyse Numér. Théor., Approx., **1** (1990), 85-91.
19. Tigan S.,. *On duality for generalized pseudomonotonic programming*, Analyse Num. Théor. Approx., **20**, *1-2* (1991), 111-116.
20. Tigan S., *On Kortanek-Evans optimality conditions for symmetric pseudoconcave programming*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca University, 1992.
21. Tigan S., *Sur le problème de la programmation vectorielle fractionnaire*, Analyse Numér. Théor. Approx., **4**, *1* (1975), 99-103.
22. Tigan S., *Linearization procedure and Kortanek-Evans optimality conditions for symmetric pseudo-concave programming*, Analyse Num. Théor. Approx., **22**, *1* (1993), 113-120.
23. Tigan S., *Optimality conditions for symmetric generalized convex programming and applications*, Studii şi Cerc. Mat., **46**, *4* (1994).
24. Weber R., *Pseudomonotonic Multiobjective Programming*, Discussion Papers B8203, Institute of Operations Research, Univ. of Saarland, Saarbruecken, 1982.
25. Weir T., Mond B., *Sufficient optimality conditions and duality for a pseudoconvex minimax problem*, Cahiers du CERO, **33**, *1-2* (1991), 123-128.

Received 1.X.1994

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