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# DIRECT METHODS FOR SOLVING SINGULAR INTEGRAL EQUATIONS ON CLOSED SMOOTH CONTOUR IN SPACES $L_p$

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There is a sufficiently large number of works concerning the foundation of the direct methods of solving singular integral equations S.I.E.) (see, for example, [1-3] and references in them). But in this works S.I.E. are considered on the standard contour (unitary circle with the centre in zero or the segment of a straight line or the real axis; the case of other contours is not enough researched. Note that the attempt to ground the direct methods for solving S.I.E. using the reduction to the standard contour leads, for example, to the loss of the smoothness of coefficients of equations which changes for the worse estimation of the rate of convergence of the method. Moreover, this way complicates essentially the computing scheme of the method and in the case of mechanical quadratures method, generally speaking, makes it impossible.

Below we propose computing schemes for the collocation and mechanical quadrature methods and give the theoretical foundation in Lebesque spaces $L_p$ for S.I.E. given on closed contour satisfying some smooth conditions without reduction of these equations to the unitary circle. In this case there appears the necessity of obtaining a series of new results from the general theory of approximate methods and approximation theory of complex variable function, given closed smooth contours, by the norm of $L_p$.

## 1. COMPUTER SCHEMES OF COLLOCATION AND MECHANICAL QUADRATURE METHODS

Let $\Gamma$ be a smooth Jordan boundary bounding a simply connected domain containing the point $t = 0$. Let $z = f(w)$ be a function which maps conformly $\{|w| > 1\}$ on the exterior of $\Gamma$ such that $\psi(\infty) = \infty$, $\psi'(\infty) = 1$.

In the complex space $L_p(\Gamma)$ $(1 < p < \infty)$ of functions $g(t)$ with the norm

$$\|g\| = \left(\frac{1}{\ell}\int_{\Gamma} |g(\tau)|^p \, |d\tau|\right)^{1/p}, \tag{1}$$

there $\ell$ is the length of $\Gamma$, consider the complete S.I.E.

$$(Mx \equiv) A(t)\ (Px)\ (t) + B(t)\ (Qx)\ (t) + \tag{2}$$
$$+\frac{1}{2\pi i}\int_{\Gamma} K(t,\tau)x(\tau)d\tau = f(t),\ t\in\Gamma,$$

where $A(t)$, $B(t)$ and $K(t,\tau)$ (the last relative to both variables) are functions with elements from $C(\Gamma)$, $f(t)$ is a function from $C(\Gamma)$, $P = (I+S)/2$, $Q = (I-P)/2$, $I$ is the identity and $S$ is the singular (with Cauchy kernel) operators, $x(t)$ is an unknown function. It is well known [4] that if $\Gamma$ is an arbitrary closed smooth contour, then $P$ and $Q$ are bounded operators in $L_p(\Gamma)$ and hence so is tho operator $M$.

According to the collocation method, the approximate solution of (2) is seeking for the polynomial

$$x_n(t) = \sum_{k=-n}^{n} \alpha_k^{(n)} t^k,\ t\in\Gamma, \tag{3}$$

the coefficients of which are unknown complex numbers $\alpha_k^{(n)} = \alpha_k$ $(k=\overline{-n,n})$;

we find these coefficients from the system of linear algebraic equations (S.L.A.E.):

$$A(t_j)\sum_{k=0}^{n}\alpha_k t_j^k + B(t_j)\sum_{k=-n}^{-1}\alpha_k t_j^k + \frac{1}{2\pi i}\sum_{k=-n}^{n}\alpha_k\int_{\Gamma}K(t_j,\tau)\tau^k d\tau = \tag{4}$$
$$= f(t_j),\ (j=\overline{0,2n}),$$

where $t_j$ $(j=\overline{0,2n})$ is a set of pairwise distinct points on $\Gamma$.

If equation (1) is solved by the mechanical quadrature method, then we find the coefficients $\alpha_k$ $(k=\overline{-n,n})$ from S.L.A.E.

$$A(t_j)\sum_{k=0}^{n}\alpha_k t_j^k + B(t_j)\sum_{k=-n}^{-1}\alpha_k t_j^k + \sum_{k=-n}^{n}\alpha_k\sum_{s=0}^{2n}k(t_j,t_s)t_s\Lambda_k^{(s)} = \tag{5}$$
$$= f(t_j)\ (j=\overline{0,2n}),$$

in which the number $\Lambda_k^{(s)}$ are determined from the relations

$$\ell_s(t) \overset{\mathrm{df}}{=} (t_s\cdot t^{-1})^n \prod_{\substack{r=0\\ s\neq r}}^{2n}\frac{t-t_r}{t_s-t_r} \equiv \sum_{k=-n}^{n}\Lambda_k^{(s)}t^k,\ t\in\Gamma. \tag{6}$$



THEOREM 1. *Let the following conditions be fulfilled:*

1) *the Riemann function* $z=\psi(w)$ *is continuously differentiable in* $\{|w|>1\}$ *twice and* $\psi''(w)$ *satisfies on* $|w|=1$ *the Hölder condition with some exponent* $\mu\in[0,1)$ *(afterwards we shall say that* $\psi$ *belongs to the class* $C(2,\mu)$*);*

2) *the functions* $A(t)$ *and* $B(t)$ *satisfy on* $\Gamma$ *the Hölder condition with exponent* $\alpha\in(0,1]$;

3) $A(t)\cdot B(t)\neq 0,\ t\in\Gamma$;

4) $ind\ A(t)\cdot B^{-1}(t) = 0,\ t\in\Gamma$;

5) $f(t)$ *and* $K(t,\tau)\in C(\Gamma)$;

6) $\dim \operatorname{Ker} M = 0$;

7) *the points* $t_j$ $(j=\overline{0,2n})$ *form a system of Fejer nodes on* $\Gamma$:

$$t_j = \psi(w_j),\ w_j = \exp\frac{2\pi i}{2n+1}(j-n),\ j=\overline{0,2n}. \tag{7}$$

*Then for all n, beginning with some* $n_i$ $(n\geq n_i)$ *the* S.L.A.E. (5) *has a unique solution* $\alpha_k$ $(k=\overline{-n,n})$. *The approximate solutions (3) converge as* $n\to\infty$ *by the nourm (1) to the exact solution of (2) with the rate*

$$\|x-x_n\| = O(n^{-\alpha}) + O\left(w\left(f;\frac{1}{n}\right)\right) + O\left(\omega^t\left(k(t,\tau);\frac{1}{n}\right)\right), \tag{8}$$

*where* $\omega\left(f,\frac{1}{n}\right)$ *and* $\omega^t\left(k(t,\tau);\frac{1}{n}\right)$ *are the continuity moduluses of f and* $K(t,\tau)$ *(relative to the variable t).*

THEOREM 2. *Let all the conditions of the theorem 1 be fulfilled. Then for* $n\geq n_2\geq n_1$ *all the assertions of theorem 1 are true chainging there* S.L.A.E. (5) *by* S.L.A.E. (6) *and adding in (8) the summand* $O\left(\omega^\tau\left(K(t,\tau);\frac{1}{n}\right)\right)$.

## 2. AUXILIARY SENTENCES

The proof of the theorem is essentially based on a series of results of constructive theory of complex variable functions, which will be obtained below for the functions, determinated on the curves of the class $C(2;\mu)$ (estimation of norm of the integration operator, estimation of derivation of interpolating Lagrange polynomial from its generating function, a.o.) as well as on some theorems about the belonging of elements to the lineal of convergency on unbounded projectors.

**2.1.** Let $U_n$ be the operator which maps any function $g(t) \in C(\Gamma)$ onto its interpolating Lagrange polynomial:

$$(U_n g)(t) = \sum_{j=0}^{2n} g(t_j)\ell_j(t), \; t \in \Gamma$$

on points $t_j$ $(j = \overline{0,2n})$; functions $\ell_j(t)$ are determined in (6).

THEOREM 3. *Let* $\Gamma \in C(2, \mu)$ *and* $\{t_j\}_0^{2n}$ *are computed acording to (7). Then*

$$\|U_n\|_{C \to L_p} \le m_1(p), 1 < p < \infty. \tag{9}$$

THEOREM 4. *For every function* $g(t) \in C(\Gamma)$ *the inequality*

$$\| U_n g - g \| \le \| I - U_n \|_{C \to L_p} E_n(g, \Gamma), \tag{10}$$

*holds, where* $E_n(g, \Gamma)$ *is the best uniform approximation of* $g(t)$, $t \in \Gamma$ *by polynomial of the form* (3).

The proof of theorems 3 and 4 is done on the basis of a series of statements from the constructive theory of complex variable functions.

**2.2.** Here theorems of belonging of elements to the lineal of convergency are established in the case when the sequence of projectors by which the approximate method is constructed consists of unbounded (on norm of the basic space) projectors.

Let $X$ be a Banach space with the norm $\|\cdot\|_x$ and $\{P\}$, $n = 1,2,\ldots$ a sequence of, generally speaking, unbounded projectors in $X$. Suppose that $A: X \to X$ is a linear bounded operator and that for $n \ge n_3$ the operators $P_n A P : X_n \to X_n$, $X_n = P_n X$ are inversible. Let us denote by $\mathscr{L}(A, P_n)$ the set of elements $y \in X$ such that, the sequence $x_n = (P_n A P_n)^{-1}$ converges by the norm of $X$ to some elements $x \in X$, the equality $Ax = y$ holds.

The set $\mathscr{L}(A, P_n)(\subset \operatorname{Im} A)$ is called the lineal of convergence of $A$ on the system of projectors $P_n$.

THEOREM 5. *Let the following conditions be fulfilled:*

1) *operator* $A: X \to X$ *is inversible and* $A^{-1}: X \to X$;

2) *for* $n \ge n_3$ *operators* $P_n A P_n : X_n \to X_n$ *are inversible and*

$$\left\|(P_n A P_n)^{-1}\right\|_{x_n} \le r_0 < \infty;$$

3) *Banach space Z is continuosly embedded in* $X: Z \subset X$ *and* $\|\cdot\|_x < r_1 \|\cdot\|_z$ *and Z is invariant relative to A;*



4) $P_n: Z \to X$ *and* $\|P_n\|_{Z \to X} \le r_2 (< \infty)$; $P_n: X \to X$ *are unbounded;*

5) $\|A - P_n A P_n\|_{x_n} \le r_s \mathscr{E}_n$, $\lim\limits_{n \to \infty} \mathscr{E}_n = 0$, $(\mathscr{E}_n > 0)$.

*Then for every* $y \in Z$ *and every* $y_n \in X_n$

$$\left\|A^{-1}y - (P_n A P_n)^{-1} P_n y\right\|_x \le \left\|A^{-1}\right\| \{(r_1 + r_2)\| y - y_n \|_z + r_0 r_2 r_3 \mathscr{E}_n \|y\|_z\}.$$

*If for every* $y \in Z$ *exists* $y_n^* \in x_n$, *then:*

$$\left\|y - y_n^*\right\|_z = \inf_{y_n \in x_n} \|y - y_n\|_z \overset{\text{df}}{=} E_n(y) \text{ and } \lim_{n \to \infty} E_n(y) = 0 \tag{10'}$$

*then* $Z \subset \mathscr{L}(A, P_n)$ *and* $\forall y \in Z$

$$\left\|A^{-1}y - (P_n A P_n)^{-1} P_n y\right\|_x \le r_4 E_n(y) + r_5 \mathscr{E}_n \|y\|_z .$$

THEOREM 6. *Let conditions* 1)-5) *of theorem* 5 *be fulfilled. If operator B satisfies the conditions*

6) *operator* $\tilde{A} = A + B$ *is inversible;*

7) $\left\| \left[(P_n A P_n)^{-1} P_n - A^{-1}\right] B \right\|_x \le r_7 \delta_n$; $\lim\limits_{n \to \infty} \delta_n = 0$,

*then for* $n \ge n_1 (\ge n_0)$ *operator* $P_n \widetilde{A} P_n$ *is inversible in* $X_n$ *and*

a) $\left\| \left(P_n \widetilde{A} P_n\right)^{-1} \right\|_{x_n} \le r_8 < \infty$;

b) $\mathscr{L}(A, P_n) = \mathscr{L}(\widetilde{A}, P_n)$;

$$\text{c) } \left\|\widetilde{A}^{-1}y - \left(P_n \widetilde{A} P_n\right)^{-1} P_n y\right\|_x \le r_0 \left\|(P_n A P_n)^{-1} P_n y - A^{-1}y\right\|_x + r_{10}\delta_n \| y \|_z . \tag{12}$$

Let $A_n$ $(n = 1,2,\ldots)$; $P_n X \to P_n X$ and suppose that for $n \ge n_4$, $A_n$ are inversible. Denote by $\widetilde{\mathscr{L}}(A, P_n)$ the set of vectors $y \in X$ having two properties:

1) the sequence $A_n^{-1} P_n y$ converges by the norm of $X$ to some element $x$ $(\in X)$;

2) the equality $Ax = y$ is true.

In the case of $A_n = P_n A P_n$ the definition of $\mathscr{L}(A, P_n)$ coincides with the definition of lineal of convergency $\mathscr{L}(A, P_n)$.

Solvability and convergence of the approximate method, constructed by a system of operators which are not projectors, is established by:

THEOREM 7. *Let conditions* 1) -5) *of theorem* 5 *be fulfilled and let the operator* $A_n : P_nX \to P_nX$ *satisfy inequality*

$$\|(P_nAP_n - A_n)z_n\|_x \le r_{11}\eta_n\|z_n\|_x, \; z_n \in P_nX. \tag{12}$$

*where* $\lim_{n\to\infty} \eta_n = 0$. *Then for* $n \ge n_3$ *the operators* $A_n : P_nX \to P_nX$ *are inversible and*

a) $\|A_n^{-1}\|_{x_n} \le r_{12}$;

b) *for every* $y \in Z$ *and every* $y_n \in X_n$ *the inequality*:

$$\|A^{-1}y - A_n^{-1}y_n\|_x \le (r_0r_2 + r_1r_{12} + r_1r_0r_{11}r_{12}\eta_n)\|y - y_n\|_z + \\ + r_0r_{12}\|y\|_z\eta_n + r_5\mathscr{E}_n\|y\|_z. \tag{13}$$

*holds.*

*If for every* $y \in Z$ *the left hand side in (10') exists and*

$\lim_{n\to\infty} E_n(y) = 0$, then $Z \subset \mathscr{L}(A, P_n)$.

The proofs of theorems 5, 6 and 7 are not difficult, they are obtained by the well-known methods of functional analysis.

### 3. THE FOUNDATION OF COMPUTER SCHEME

The S.L.A.E. (3) is equivalent to the operator equation

$$(M_nx_n \equiv)U_nMx_n = U_nf. \tag{14}$$

considered as an equation in the space $X_n$ of polynomial of the form (3) with norm (1). By conditions 2) – 4) the function $B^{-1}(t)A(t)$ admits the canonic factorization $B^{-1}(t)A(t) = C_+(t)C_-(t)$. $C_+^{\pm 1} \in PH_\alpha$, $C_-^{\pm 1}(t) \in QH_\alpha + \{\text{const}\}$. Obviously, (14) is equivalent to the operator equation

$$(R_nx_n \equiv)U_n\left[PC_- + QC_+^{-1} + QC_-P + PC_+^{-1}Q + C_+^{-1}B^{-1}T\right]x_n = \\ = U_n\left[C_+^{-1}(t)B^{-1}(t)f(t)\right],$$

$T$ is the integral operator with the kernel $K(t, \tau)$.



Let $C_-^{(n)}(t)$, $C_+^{(n)}(t)$ be the polynomial functions of the best uniform approximation of functions $C_-(t)$, $C_+^{-1}(t)$ respectively. Then for sufficiently large $n$ $C_\pm^{(n)}(t) \ne 0$, $t \in \Gamma$ and by stability of index for the functions $C_\pm^{(n)}(t)$ the indices are equal to zero. Therefore for such $n$ the operator

$$R_n^{(1)} = U_n\left[PC_-^{(n)} + QC_+^{(n)}\right]U_n$$

is inversible in $X_n$ and $\left\|\left(R_n^{(1)}\right)^{-1}\right\|_{x_n} = O(1)$.

Using now theorems 3, 4 it is easy to find that

$$\left\|R_n^{(1)} - R_n^{(2)}\right\|_{x_n} = O(n^{-\alpha}), \text{ where } R_n^{(2)} = U_n[PC_- + QC_+^{-1}]U_n.$$

Then by Banach theorem for sufficient large $n \ge n_0$ operator $R_n^{(2)}$ is inversible in $[X_n]$, and $\left\|\left(R_n^{(2)}\right)^{-1}\right\|_{x_n} = O(1)$.

Using now theorems 5, 6, we get that if conditions 1), 5)-7) are fulfilled, then operator $R_n$ and together with it also $M_n$ for sufficiently large $n \ge n_1$ is inversible in $X_n$ and $\|M_n^{-1}\|_{x_n} = O(1)$.

So the solvability of S.L.A.E. (4) is established.

The estimation of the rate of convergence follows from (11). Theorem 1 is proved.

Let us prove now theorem 2. S.L.A.E. (5) is equivalent to the following operator equations

$$(F_nx_n \equiv)U_n[PC_- + QC_+^{-1} + QC_-P + PC_+^{-1}Q + \Delta_n]x_n = U_nf_1, \tag{15}$$

where $\Delta_n$ is an operator defined by the formula

$$\Delta_nx_n = C_+^{-1}(t)B^{-1}(t)\frac{1}{2\pi i}\int_\Gamma \frac{1}{\tau}U_n^\tau[\tau K(t,\tau)]x_n(\tau)d\tau, \; t \in \Gamma;$$

$$f_1(t) = C_+^{-1}(t)B^{-1}(t)f(t).$$

Let us verify the fulfilment of conditions of theorem 7, putting there

$$X = L_p(\Gamma),\ 1 < p < \infty,\ z = C(\Gamma),\ P_n = U_n,$$
$$A = V + K_1 + K_2,\ A_n = F_n(= U_n(V + K_2 + \Delta_n)U_n).$$

Conditions 1)–5) of theorem 5 have been verified when theorem 1 was proved. So verify condition (12) and relation $\lim_{n\to\infty}\eta_n = 0$. Let $z_n(t) \in \mathcal{P}_n$. Then

$$\gamma^{(1)} \overset{df}{=} \|(P_n A P_n - A_n)z_n\|_p = \|[U_n(V + K_1 + K_2)U_n - U_n(V + K_2 + \Delta_n U_n)]z_n\|_p =$$

$$= \|U_n(K_1 - \Delta_n)z_n\|_p = \left\| U_n\left\{\frac{1}{2\pi i}\int_\Gamma K(t,\tau)z_n(\tau)d\tau - \right.\right.$$

$$\left.\left. -\frac{1}{2\pi i}\int_\Gamma \frac{1}{\tau}U_n^\tau[\tau K(t,\tau)]x_n(\tau)d\tau\right\}\right\|_p =$$

$$= \left\| U_n\left\{\frac{1}{2\pi i}\int_\Gamma \frac{1}{\tau}\left\{\tau K(t,\tau) - U_n^\tau[\tau K(t,\tau)]\right\}z_n(\tau)d\tau\right\}\right\|_p .$$

Using theorem 3 we get

$$\gamma_n^{(1)} \le m_1(p)\left\|\frac{1}{2\pi i}\int_\Gamma \frac{1}{\tau}\{\tau K(t,\tau) - U_n^\tau[\tau K(t,\tau)]\}z_n(\tau)d\tau\right\|_{c'}$$

hence by Holder inequality for integrals we find

$$\gamma_n^{(1)} \le m_1(p)\frac{1}{2\pi}c_2\left\|\tau K(t,\tau) - U_n^\tau[\tau K(t,\tau)]\right\|_p \|z_n\|_p .$$

From here and from Jakson theorem we obtain

$$\gamma_n^{(1)} \le c(1 + m_1(p))E_n^\tau(\tau K(t,\tau);\Gamma) \le c(1 + m_1(p))\frac{1}{n^r}\omega^\tau\left(K^{(r)};\frac{1}{n}\right).$$

From the definition of $\gamma_n^{(1)}$ and the last estimation one can see that condition (12) and $\lim_{n\to\infty}\eta_n = 0$ are fulfilled and

$$\eta_n = \omega^\tau(K^{(r)};1/n)/n^r.$$

Thus it is proved that all conditions of theorem 7 are fulfilled. Then by this theorem and inequality (13) the following are true:

1) beginning with numbers $n \ge n_1$ operator $F_n = U_n(V + K_1 + \Delta_n)U_n$ is inversible in $[X_n]$ and $\|F_n^{-1}\| = O(1)$; so equation (15) and S.L.A.E. (5) together with it are uniquely solvable;



2) for every function $g(t) \in [C(\Gamma)]$ are fulfilled the inequality

$$\|(V + K_1 + K_2)^{-1}g - F_n^{-1}U_n g\|_p \le (c_1 + c_2\eta_n)\cdot E_n(g;\Gamma) + c_3\|g\|_c \eta_n +$$
$$+c_4\frac{1}{n^{r+\alpha}}\|g\|_c + c_5\frac{1}{n^r}\omega\left(f^{(r)};\frac{1}{n}\right) + c_6\cdot\frac{1}{n^r}\omega^t\left(K(t,\tau);\frac{1}{n}\right); \qquad (16)$$

3) $[C(\Gamma)] \subset \widetilde{\mathscr{L}}(F_n, U_n)$.

Putting $g(t) = f_1(t)$ in (16) and taking into acount that

$$(V + K_1 + K_2)^{-1}f_1 = x(t),\ F_n^{-1}U_n f_1 = x_n$$

and $f_1(t) \in [C^r(\Gamma)]$ by Jakson theorem we obtained the demanded estimation. Theorem 2 is proved.

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