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BOOK REVIEWS

VILMOS TOTIK, *Weighted Approximation with Varying Weight.* Lecture Notes in Mathematics, 1569. Springer-Verlag, Berlin, Heidelberg, 1994, 115 pp., ISBN 3-540-57705-X

In this book the problem of polynomial approximation with weighted polynomials of the form $w^n P_n$, where w is some fixed weight (e.g. Freud, Jacobi or Laguerre weights), and the degree of P_n is at most n, is generalized for a large family of weights. The generalization given here is the first general result in the subject, and is far stronger than the presently existing results. It also solves several open conjectures. This method is extended to the case of varying weights in the stronger sense, and the results obtained herein are used in the discussion of some applications.

The book contains a short Introduction (5 pages); 4 chapters: I. Freud weights, II. Approximation with general weights, III. Varying weights, IV. Applications; References and Index.

The first chapter contains a presentation of the results concerning the approximation problem for Freud weights (exponential type weights) and the strong asymptotic result of Lubinsky and Saff for an extremal problem associated with Freud weights.

In the second chapter the previous results are generalized and a solution is given for the analogous approximation problem for a large family of weights, namely for admissible weight functions w on $\sum$, where $\sum$ is a regular closed subset of the real line. The method used herein consists in the approximation of a given potential by first translating the generating measure and then appropriately discretizing it. Two important examples are here constructed.

In the third chapter there is a study of the problem of the approximation by weighted polynomials $w_n^n P_n$ with varying weights. The results from this chapter are obtained either by direct application of the results from the previous chapter, either by the modification of the above method. The idea here is similar, but instead of translation one uses projection onto a curve, which is closer to the support exactly where the generating measure is larger.

The fourth chapter contains some interesting applications of the previous results to various approximation problems with varying weights.

To conclude, we may say that the present book contains new and original theoretical results as well as applications in the field of polynomial approximation with weighted polynomials.

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WU LI, *Continuous Selections for Metric Projections and Interpolating Subspaces*, Peter Lang, Frankfurt-Bern-New York-Paris, 1991, 108 pp.

The aim of the present book is to study the existence of continuous selections for the metric projection P_G onto a finite dimensional subspace G of the Banach space $C_0(T)$. As usual, for a locally compact Hausdorff space T one denotes by $C_0(T)$ the Banach space (with respect to the sup-norm) of all continuous real-valued functions on T, vanishing at infinity.

By a classical result of A. Haar and A.N. Kolmogorov an n-dimensional subspace $G = sp\{g_1, ..., g_n\}$ of $C(T)$ (T - compact) is a Chebyshev subspace iff it is a Haar subspace, i.e. each $g \in G \setminus \{0\}$ has at most $(n-1)$ zeros in T. A system $\{g_1, ..., g_n\} \subset C(T)$ satisfying this condition is called also a Chebyshev system (see e.g. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Publishing House of the Romanian Academy and Springer Verlag, Bucharest and Berlin 1970, p.182). By a conjecture of S. Mazur proved by J.C. Mairhuber, K. Sieklucki and P.C. Curtis (see the above quoted book of I. Singer, p.219) the existence of a Haar subspace G of $C(T)$ of dimension $n \geq 2$ forces T to be homeomorphic to a subset of the unit circle $T \subset R^2$.

In order to give intrinsic characterizations for the existence of a continuous selection for the metric projection P_G, the author considers the more general notion of a quasi-Haar subspace: a finite dimensional subspace G of $C_0(T)$ is called quasi-Haar provided card $(bdZ(g)) \leq \dim N(int\, Z(g))\ (=: r_g)$ and g has at most $(r_g - 1)$ zeros with sign changes, for every $g \in G \setminus \{0\}$. A zero with sign changes is an element $t \in T$ such that $g(t) = 0$ and g takes both positive and negative values in any neighborhood of t. Also, for $g \in C_0(T)$, let $Z(g) = \{t \in T : g(t) = 0\}$, $int\, Z(g)$ and $bdZ(g)$ be the interior respectively the boundary of $Z(g)$. For $A \subset T$ and $B \subset G$ let $N(A) = \{g \in G : A \subset Z(g)\}$ and $Z(B) = \bigcap \{Z(g) : g \in B\}$. Using this notion, the author proves that if P_G admits a continuous selection, then G must be a quasi-Haar subspace, of $C_0(T)$ (Theorem 1.3). The converse is true only under the additional hypothesis that the locally compact space T is also locally connected (Theorem 1.4).

An n-dimensional subspace G of $C_0(T)$ is called weakly interpolating if for any set of n points $\{t_i\}_1^n \subset T$ and any $\{\varepsilon_i\}_1^n \subset \{-1,1\}$ there exist $g \in G \setminus \{0\}$ and neighborhoods V_i of t_i such that $\varepsilon_i g(t) \geq 0$, for all $t \in V_i$, $i = 1, ..., n$. A subspace G of $C_0(T)$ is called a Z-subspace if int $Z(g) = \emptyset$, for every $g \in G \setminus \{0\}$. The metric projection P_G onto on n-dimensional Z-subspace G of $C_0(T)$ has a continuous selection if and only if G is a weakly interpolating subspace and every $g \in G \setminus \{0\}$ has at most n zeros (Theorem 10.2).

In order to obtain a full characterization of finite dimensional subspaces G of $C_0(T)$ for which the metric projection P_G has a continuous selection the author introduces a stronger notion, namely that of regular weakly interpolating subspace (see Definition 1.2 in the book). Then, a finite dimensional subspace G of $C_0(T)$ is regular weakly interpolating if and only if the metric projection P_G admits a continuous selection (Theorems 1.5, 3.1 and 10.1).

One of the basic notions used in the proofs of the above-mentioned theorem is that of extremal signature considered by T. Rivlin, H. Shapiro, B. Brosowski and the author. The second section of the book, entitled Extremal Signatures, is devoted to the study of various properties of extremal signatures. The first section, called Introduction, contains a detailed survey of the results proved in the book. The rest of the chapters are headed as follows: 3. Invariance of Regular Weakly Interpolating Subspaces; 5. Semidefinite Property of Regular Weakly Interpolating Subspaces; 6. Two Perturbation Theorems; 7. Alternation Signatures; 8. Local Alternation Elements; 9. Strict Best Approximation; 10. Characterization of Continuous Metric Selections; 11. Nürnberger and Sommer's Characterization Conditions.

Written by an eminent specialist in the field and largely based on his original results, the book is a valuable contribution to this domain of research. It is worth mentioning that some of the results included in this book were previously published by the author in Scientia Sinica and Acta Mathematica Sinica (some of them even in Chinese), so that the present book supplies the mathematical community with a more accessible way to them.

The book is clearly written, contains a wealth of results, covering very well the proposed theme and, undoubtedly, will become an indispensable tool for all people working in abstract approximation theory and related areas.

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ANATOLIJ ANTONEVICH, *Linear Functional Equations. Operator Approach, Operator Theory. Advances and Applications*, vol.83, Birkhäuser Verlag, Basel-Boston-Berlin 1996, 179 pp.

The present book is a translation (given by V. Muzafarov and A. Iacob) of the Russian edition published by the University of Minsk in 1988. Its aim is to study operators of the form

$$bu(x) = \sum_{k=1}^{m} a_k(x) u(\alpha_k(x)), \tag{1}$$

acting on a given space of functions $F(X)$. Here α_k are mappings from X to X and a_k are functions on X. The operators of the form (1) are called functional operators and their study concerns mainly properties needed for solving the functional equation

$$bu(x) = f(x), \tag{2}$$

such as spectral properties, invertibility, Fredholm and Noether properties, etc. Usually $F(X)$ is a space of integrable, continuous or differentiable functions.

Obviously this general scheme incorporates a lot of particular problems arising in various areas of mathematics as functional equations, functional-differential equations with deviating argument, nonlocal problems for partial differential equations, problems in general theory of Banach algebras, dynamical systems and complex function theory. The book is based mainly on results concerning Banach algebras generated by functional operators, a direction of research initiated by the author at Minsk University. This general approach emphasizes unexpected connections between apparently unrelated previously known results and, at the same time, allows to obtain new results. One should mention an other related book by the author and Ja. V. Radyno, *Functional Analysis and Integral Equations*, Minsk University Press, 1988 (in Russian).

The book is divided into six chapters. The first one, Chapter 0, contains a survey of some notions and results from functional analysis, measure theory, dynamical systems, vector bundles and differentiable manifolds, needed in the rest of the book.

Ch.1, Functional Operators, is concerned with the study of the simplest case of functional operators, namely operators of the form $T_\alpha u(x) = u(\alpha(x))$ and $bu(x) = a(x)\, u(\alpha(x))$ called shift operators and respectively weighted shift operators (other terms used for these operators are substitution operators, composition operators, etc.). The author gives some conditions on the functions a and α ensuring the boundedness or invertibility of these operators. The spectrum and spectral radius are also studied.

In Ch.2, Banach Algebras Generated by Functional Operators, the author proposes an axiomatic approach to some algebraic properties of functional operators (Noether property, the index, spectrum, etc.). The general framework is that of a Banach algebra B and a representation T of a group G in B. When B is a C^*-algebra one supposes further that the representation T is unitary. This axiomatic approach is inspired from the study of functional operators given in Ch. 1, by isolating some properties of model operators. This general study is continued in Ch.3, Invertibility Conditions for Functional Operators. L_2 - Theory.

In Ch. 4, Functional Operators in Some Special Function Spaces, the general theory is applied to the study of functional operators and equations in L_p-spaces and in spaces of continuous functions. Functional equations in the space of periodic distributions are also considered. Other interesting and non-trivial applications are given in Ch. 5, Applications to Some Classes of Equations and Boundary Value Problems.

The main text ends with a section of comments of historical character, bibliographical information and references for further reading. A list of references (258 items) and a notion index are also included.

The book is clearly written, contains many interesting results, providing a unified approach to a variety of relatively dispersed results. It will be useful to functional analysts looking for applications as well as for mathematicians interested in various kinds of equations, providing them with powerful tools of investigation.

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