REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION

Tome 25, N^{os} 1–2, 1996, pp. 23–31

THE APPLICATION OF KANTOROVICH-RITZ METHOD TO THE FLOW OF AN INCOMPRESSIBLE POTENTIAL FLUID BETWEEN TWO SOLID WALLS

DOINA BRĂDEANU

(Cluj-Napoca)

1. THE FORMULATION OF THE PROBLEM

a) *Differential problem.* Consider the plane Oxy and the bounded domain $\overline{\Omega} = \{(x, y) \in \mathbf{R}^2 | -x_0 \le x \le x_0,\ 0 \le y \le y_w(x)\}$, where $y = y_w(x)$ is the given equation of the $\overline{CB}$ boundary ,fig 1. It is assumed that $\overline{DA}$ and $\overline{CB}$ boundaries are solid walls and that the ideal incompressible fluid between them irrotationally flows in the positive x direction with the velocity $\vec{V}(x,y) = (V_x, V_y)$. On $\overline{DC}$ and $\overline{AB}$ boundaries the velocity vector is constant and parallel to Ox axis.

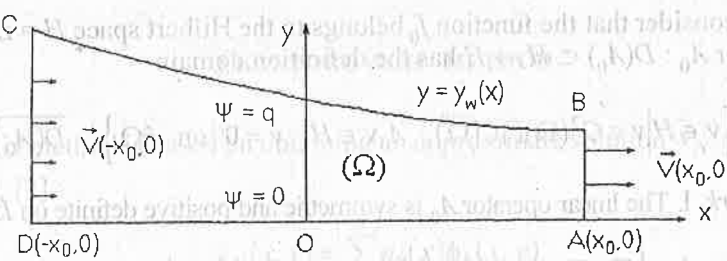

Fig. 1

With these assumptions, the stream function ψ satisfies Laplace's equation on Ω, $\Delta\psi=0$.

The boundary conditions on the solid walls are $\psi(x, 0) = \psi_0 = 0$ and $\psi(x,y_w(x))=$ $= \psi_w(x) = q$, where q is a positive constant. It is known from the definition of the stream function that $V_x(-x_0, y) = (\partial\psi / \partial y)_{x=-x_0}$ = const.$\equiv k_0^-$ and using the relation $\psi(-x_0,0)=0$, we obtain that $\psi(-x_0,y)=k_0^- y$ and similarly, $\psi(x_0,y)=k_0^+ y$. The flow rate in the x direction being constant, we have [2]

$$\int_0^{y_w(-x_0)}\frac{\partial\psi}{\partial y}(-x_0,y)\mathrm{d}y=\int_0^{y_w(x_0)}\frac{\partial\psi}{\partial y}(x_0,y)\mathrm{d}y=q$$

Hence ψ satisfies the boundary conditions

(1) $$\psi(\pm x_0,y)=\frac{qy}{y_w(\pm x_0)},\ 0\le y\le y_w$$

Then the transformation

(2) $$\psi(x,y)=\frac{qy}{y_w(x)}-v(x,y)$$

is made in equation $\Delta\psi=0$. The problem reduces to solving the following boundary value problem with homogeneous Dirichlet boundary conditions :

(3) $$\Delta v(x,y)=qy\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{1}{y_w(x)},\ (x,y)\in\Omega$$

(4) $$v(x,0)=0,\ v(x,y_w(x))=0,\ v(-x_0,y)=0,\ \ v(x_0,y)=0$$

b) *Equivalent variational problem.* The boundary value problem can now be written in the form of operatorial equation, i.e.

(5) $$A_0v=f_0$$

$$\left(f_0=-qy\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{1}{y_w(x)},\ \ A_0=-\Delta=-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)$$

where we consider that the function f_0 belongs to the Hilbert space $H=L_2(\Omega)$ and the operator $A_0:D(A_0)\subset H\to H$ has the definition domain

$$D(A_0)=\left\{v\in H\middle|v\in C^2(\Omega)\cap C(\overline{\Omega}),\ A_0v\in H,\ \ v=0\ \text{ on }\ \partial\Omega\right\};\ \ \overline{D(A_0)}^H=H$$

Remark 1. The linear operator A_0 is symmetric and positive definite on $D(A_0)$, [5]:

$$(A_0u,v)=\int_\Omega\nabla u\cdot\nabla v\mathrm{d}\Omega=(u,A_0v),(A_0v,v)\ge0,\ \ (A_0v,v)=0\Leftrightarrow v=0;$$

$$\exists\alpha^2>0\ \text{ such that }\ (A_0v,v)\ge\alpha^2(v,v),\ \ \alpha^2=\frac{1}{C_F(\Omega)}$$

where $C_F(\Omega)$ is the constant in the Friedrichs inequality and $(.,.)$ denotes the scalar product in $L_2(\Omega)$.

The energetic space H_{A_0} of the operator A_0 can be identified with the Sobolev space $H_0^1(\Omega)$ endowed with energetic norm and energetic scalar product, respectively [6]:

$$H_{A_0}=H_0^1(\Omega)=\{v\in H^1|v=0\ \text{on}\ \partial\Omega\,(\text{in the trace sense})\}$$

$$\|v\|_{A_0}^2=\int_\Omega|\nabla v|^2\mathrm{d}\Omega;\ \ (u,v)_{A_0}=\int_\Omega\nabla u\cdot\nabla v\mathrm{d}\Omega$$

– According to the properties from Remark 1 and using the theorem of the minimum of the energy functional $F_0(v)$ on H_{A_0}, the operatorial equation (5) is equivalent to the following variational problem [6] :

(6) $$F_0(v)\to\text{minimum}\quad\text{on}\quad D(F_0)=H_{A_0}$$

where

(7) $$F_0(v)=\|v\|_{A_0}^2-2(f_0,v)=\int_\Omega\left(|\nabla v|^2-2f_0v\right)\mathrm{d}\Omega$$

Remark 2. The solution $\tilde{v}\in H_{A_0}$ of problem (6), which exists and is unique, is the generalized solution of problem (3)-(4) and has the form

$$\tilde{v}=\sum_{k=1}^{\infty}(f_0,w_k)w_k$$

if $\{w_k\}_{k=1}^{\infty}$ is an orthonormal and complete system of functions in the energetic space H_{A_0}.

2. KANTOROVICH METHOD

This method is based on obtaining an approximate solution $v_N\in H_{A_0}$ of the form [4], [3] :

(8) $$v_N(x,y)=\sum_{k=1}^{N}a_k(x)\phi_k(x,y)$$

which is of N-order approximation for $\tilde{v}$. Here a_k, $k=1,2,\ldots,N$, are unknown functions on $[-x_0,x_0]$ and ϕ_k, $k=1,2,\ldots,N$, are known functions on Ω belonging to H_{A_0}, or to $D(A_0)$ if the considered problem can be formulated in $D(A_0)$. These functions have to be chosen so that conditions (4) for v_N be valid.

According to the Kantorovich procedure:

a) The trial functions ϕ_k, $k = 1, 2, \ldots, N$, are chosen from a complete (nonorthonormal) system of functions $\{\phi_k\}_{k=1}^{\infty}$ and satisfy the conditions $\phi_k(x, y) = 0$ on $\partial\Omega$, excluding the lines $x = \pm x_0$. Then the trial functions can be chosen in the form

(9) $$\phi_k(x,y) = [y - y_w(x)]y^k, \quad k = 1, 2, \ldots, N$$

b) The functions a_k satisfy the following system of ordinary differential equations, [1] :

(10) $$\int_0^{y_w(x)} (A_0 v_N - f_0)\phi_i(x,y)\,\mathrm{d}y = 0, \qquad i = 1, 2, \ldots, N$$

and the conditions

(11) $$a_k(-x_0) = a_k(x_0) = 0, \quad k = 1, 2, \ldots, N$$

obtained from the stationary condition of the functional F_0 on v_N written as

$$F_0(v_N) = \int_{-x_0}^{x_0} \left\{ \int_0^{y_w(x)} \left[\left(\frac{\partial v_N}{\partial x}\right)^2 + \left(\frac{\partial v_N}{\partial y}\right)^2 - 2 f_0 v_N \right] \mathrm{d}y \right\} \mathrm{d}x \equiv$$

$$\equiv \int_{-x_0}^{x_0} G(x, a_1(x), \ldots, a_N(x), a'_1(x), \ldots, a'_N(x))\,\mathrm{d}x,$$

i.e. from the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial G}{\partial a'_k} - \frac{\partial G}{\partial a_k} = 0 \quad \text{with} \quad a_k(-x_0) = a_k(x_0) = 0, \quad k = \overline{1,N}$$

By substituting A_0, v_N, f_0 and ϕ_k with their expressions and performing some calculations, the differential problem (10)-(11) becomes

(12) $$\sum_{k=1}^{N} \left[\alpha_{ik}(x)\frac{\mathrm{d}^2 a_k}{\mathrm{d}x^2} + 2\beta_{ik}(x)\frac{\mathrm{d}a_k}{\mathrm{d}x} + \gamma_{ik}(x)a_k(x) \right] = r_i(x), \quad i = \overline{1,N}$$

(13) $$a_k(-x_0) = a_k(x_0) = 0, \quad k = \overline{1,N}$$

where

$$\alpha_{ik}(x) = \int_0^{y_w(x)} \phi_i(x,y)\phi_k(x,y)\,\mathrm{d}y = \frac{2y_w^{i+k+3}(x)}{(i+k+1)(i+k+2)(i+k+3)}$$

$$\beta_{ik}(x) = \int_0^{y_w(x)} \phi_i(x,y)\frac{\partial \phi_k}{\partial x}\,\mathrm{d}y = \frac{y_w^{i+k+2}(x)y'_w(x)}{(i+k+1)(i+k+2)}$$

$$\gamma_{ik}(x) = \int_0^{y_w(x)} \phi_i \Delta\phi_k\,\mathrm{d}y = \left[\frac{y_w(x)y''_w(x)}{i+k+2} - \frac{2ik}{(i+k-1)(i+k)} \right] \frac{y_w^{i+k+1}(x)}{i+k+1}$$

$$r_i(x) = -\int_0^{y_w(x)} f_0(x,y)\phi_i(x,y)\,\mathrm{d}y = q\frac{\left[y_w(x)y''_w(x) - 2y'^2_w(x)\right]y_w^i(x)}{(i+2)(i+3)}$$

3. THE CALCULATION OF THE FIRST-ORDER KANTOROVICH APPROXIMATION

a) *The boundary value problem.* Consider that the upper solid surface is a flat plate and the equation of the line $\overline{CB}$ is

$$y_w(x) = \alpha_0 x + \beta_0, \quad x \in [-x_0, x_0]$$

In this case the boundary value problem (12)-(13) becomes

$$-\sum_{k=1}^{N} \left[\frac{(i+k)!}{(i+k+3)!}\frac{\mathrm{d}}{\mathrm{d}x}\left(y_w^{i+k+3}(x)\frac{\mathrm{d}a_k}{\mathrm{d}x} \right) - \frac{ik(i+k-2)!}{(i+k+1)!} y_w^{i+k+1}(x)a_k(x) \right] =$$

$$= q\frac{y'^2_w(x)y_w^i(x)}{(i+2)(i+3)}, \quad i = \overline{1,N}; \quad a_k(-x_0) = a_k(x_0) = 0, \quad k = \overline{1,N}$$

If we choose $N = 1$ (i.e. $i = k = 1$) and we write $x = x_0 t$, $y_w(x) = w(t) = \alpha_0 x_0 t + \beta_0$, $a_1(x) = u(t)$, $t \in (-1,1) = I$, we obtain the following ordinary differential equation and the boundary conditions:

(14) $$Au \equiv -\frac{\mathrm{d}}{\mathrm{d}t}\left(w^5(t)\frac{\mathrm{d}u}{\mathrm{d}t} \right) + 10x_0^2 w^3(t)u(t) = 5qw(t)w'^2(t), \qquad t \in (-1, 1)$$

(15) $$u(-1) = 0, \quad u(1) = 0$$

This problem can be written in the operatorial form

(16) $$Au = f,$$

with

$$A: D(A) \subset L_2(I) \to L_2(I)$$

$$D(A) = \left\{ u \in L_2(I) \,\middle|\, u \in C^2(I) \cap C(\bar{I}),\ Au \in L_2(I), u(-1) = u(1) = 0 \right\}$$

$$f(t) = 5qw(t)w'^2(t)$$

Taking into account that the operator A is linear, symmetric and positive definite on $D(A)$ and its energetic space $H_A = H_0^1(I)$ is endowed with the energetic norm $\|\cdot\|_A$ (and the energetic scalar product $(.,.)_A$), [6] ,

$$\|u\|_A^2=(u,u)_A=\int_{-1}^{1}\left(w^5u'^2+10x_0^2w^3u^2\right)dt$$

it follows that the operatorial equation (i.e. the boundary value problem (14)-(15)) is equivalent to the variational problem

(17) $$F(u)=\|u\|_A^2-10q\int_{-1}^{1}w(t)w'^2(t)u(t)dt\to \text{minimum on } H_A$$

b) *Ritz algorithm*. With given trial functions $\varphi_k(t)$, $k=\overline{1,n}$[$\{\varphi_k\}_{k=1}^{\infty}$ is a complete base in H_A], we construct the subspace $H_A^{(n)}=\text{span}\{\varphi_1,\dots,\varphi_n\}$ and we choose for the solution $\tilde{u}$ of (17) the Ritz approximate solution $u_n\in H_A^{(n)}$, [5] in the form

(18) $$u_n(t)=\sum_{k=1}^{n}c_k\varphi_k(t),\quad c_k\in \mathbf{R}^1$$

The unknown coefficients c_k, $k=\overline{1,n}$ are obtained using the Ritz procedure by solving the linear system of algebraic equations (Ritz system)

(19) $$\sum_{k=1}^{n}K_{jk}c_k=b_j\ ,\ j=\overline{1,n}$$

where

$$K_{jk}=(\varphi_j,\varphi_k)_A\ ,\quad b_j=5qw'^2(w,\varphi_j)_{L_2(I)}$$

$$\left((Au_n-f,\varphi_j)_{L_2(I)}=0=\sum_{k=1}^{n}c_k(A\varphi_k,\varphi_j)_{L_2(I)}-(f,\varphi_j)_{L_2(I)}\right)$$

Remark 3. Since $H=L_2(I)$ is a separable space and H_A is also separable, there will also exist complete systems of functions $\{\varphi_k\}_{k=1}^{\infty}$ in H_A.

- The trial functions are chosen to be of the form

$$\varphi_k(t)=\frac{1}{k\pi}\sin k\pi t\ ,\ t\in[-1,1],\ k=\overline{1,n}$$

where n is a given natural number ; $\varphi_k\in H_A$.

The coefficients of the Ritz system (19) are

$$2K_{jk}=\int_{-1}^{1}\left[w^5(t)+\frac{10x_0^2}{jk\pi^2}w^3(t)\right]\cos(j-k)\pi t\,dt+$$

$$+\int_{-1}^{1}\left[w^5(t)-\frac{10x_0}{jk\pi^2}w^3(t)\right]\cos(j+k)\pi t\,dt,\ j,k=\overline{1,n}$$

In order to calculate these coefficients and the right hand side of (19), the following exact formulas are successively obtained

$$I_m^{(5)}=\int_{-1}^{1}w^5(t)\cos m\pi t\,dt=$$

$$=\frac{(-1)^m}{m^2\pi^2}\left\{(w^5)'(1)-(w^5)'(-1)-\frac{1}{m^2\pi^2}\left[(w^5)'''(1)-(w^5)'''(-1)\right]\right\}=$$

$$=\frac{(-1)^m}{m^2}\left(\frac{\alpha_0x_0}{\pi}\right)^2 40\beta_0\left[\beta_0^2+\alpha_0^2x_0^2-\frac{6}{m^2}\left(\frac{\alpha_0x_0}{\pi}\right)^2\right],\quad m=\overline{1,2n}$$

$$I_m^{(3)}=\int_{-1}^{1}w^3(t)\cos m\pi t\,dt=\frac{(-1)^m}{m^2\pi^2}\left[(w^3)'(1)-(w^3)'(-1)\right]=$$

$$=\frac{(-1)^m}{m^2}12\beta_0\left(\frac{\alpha_0x_0}{\pi}\right)^2,\quad m=\overline{1,2n}$$

$$\left(I_0^{(s)}=\int_{-1}^{1}w^s(t)\,dt;\quad s=3;s=5\right)$$

$$b_j=\frac{5qw'^2}{j\pi}\int_{-1}^{1}w(t)\sin j\pi t\,dt=\frac{(-1)^{j+1}}{j^2}10q\pi\left(\frac{\alpha_0x_0}{\pi}\right)^3,\qquad j=\overline{1,n}$$

c) *Numerical application.* By assuming that $x_0=1$, (i.e. $x=t$), $\alpha_0=-\frac{1}{4}$, $\beta_0=\frac{3}{4}$, (i.e. $y_w(x)=-\frac{1}{4}(x-3)$), $n=3$, $q=1$ and taking into account the above expressions, we have

$$K_{11}=\frac{1}{2}\left[I_0^{(5)}+I_2^{(5)}+\frac{10}{\pi^2}\left(I_0^{(3)}-I_2^{(3)}\right)\right]=0.810466$$

$$K_{22}=\frac{1}{2}\left[I_0^{(5)}-I_4^{(5)}+\frac{5}{2\pi^2}\left(I_0^{(3)}-I_4^{(3)}\right)\right]=0.450106$$

$$K_{33}=\frac{1}{2}\left[I_0^{(5)}-I_6^{(5)}+\frac{10}{9\pi^2}\left(I_0^{(3)}-I_6^{(3)}\right)\right]=0.381254$$

$$K_{12}=\frac{1}{2}\left[I_1^{(5)}+I_3^{(5)}+\frac{5}{\pi^2}\left(I_1^{(3)}-I_3^{(3)}\right)\right]=-0.069901$$

$$K_{13}=\frac{1}{2}\left[I_2^{(5)}+I_4^{(5)}+\frac{10}{3\pi^2}\left(I_2^{(3)}-I_4^{(3)}\right)\right]=0.020117$$

$$K_{23}=\frac{1}{2}\left[I_1^{(5)}+I_5^{(5)}+\frac{5}{3\pi^2}\left(I_1^{(3)}-I_5^{(3)}\right)\right]=-0.062747$$

$$b_1 = -0.015831\ ;\ b_2 = 0.003957\ ;\ b_3 = -0.001759$$

if we consider only six decimal digits.

The solution of the Ritz system (19) for $n=3$ is

$$c_1 = 0.018195\ ;\ c_2 = 0.005465\ ;\ c_3 = -0.002711$$

and then the third order Ritz approximate solution $u_3(x)$ ($\equiv a_1(x)$) has the form

$$a_1(x) = \frac{1}{\pi}\left(c_1 \sin \pi x + \frac{1}{2} c_2 \sin 2\pi x + \frac{1}{3} c_3 \sin 3\pi x \right) \tag{20}$$

The approximation of the stream function given by Kantorovich-Ritz method is

$$\psi_a(x,y) = \left\{ \frac{1}{y_w(x)} - a_1(x)\left[y - y_w(x) \right] \right\} y \tag{21}$$

d) *Test. Error.* The explicit approximate equations $y = f(x,k)$ of the stream lines (obtained from $\psi_a(x,y) = k$, k =const., $k \in [0,1]$) in the Kantorovich-Ritz approximation are found by solving

$$a_1(x)y^2 - \left[\frac{1}{y_w(x)} + a_1(x) y_w(x) \right] y + k = 0 \tag{22}$$

Table 1.

x	$a_1(x)$	$e\left(x,\frac{1}{4}\right)$	$e\left(x,\frac{1}{2}\right)$	$e\left(x,\frac{3}{4}\right)$
−0.75	0.00535	0.00083	0.00110	0.00083
−0.50	0.00576	0.00072	0.00096	0.00072
−0.25	0.00361	0.00037	0.00048	0.00037
0.25	−0.00361	−0.00021	−0.00029	−0.00022
0.50	−0.00576	−0.00026	−0.00035	−0.00026
0.75	−0.00535	−0.00017	−0.00024	−0.00017

taking into account that $0 \le y \le y_w(x)$.

On the other hand, the exact equations of the stream lines are straight lines $y = g(x,k)$, where $g(x,k) = \frac{k}{4}(3-x)$.

The approximate stream function (21) obtained by Kantorovich-Ritz method is tested in Table 1 which contains the values $a_1(x)$ from (20) and also the errors $e(x,k) = f(x,k) - g(x,k)$ with respect to a stream line (by considering $k = \frac{1}{4}$, $k = \frac{1}{2}$, $k = \frac{3}{4}$) for several values of $x \in [-1,1]$. One can observe that these errors are small and consequently, the exact stream function in the case of two plane walls is well approximated by ψ_a given in (21).

4. CONCLUSIONS

It was shown that the Kantorovich-Ritz method can be successfully applied for the incompressible potential fluid flow between two walls. In order to calculate the approximation of the stream function we obtained :

1°. A mathematical model (3)-(4) for the stream function ψ of a steady state two-dimensional flow in a bounded domain;

2°. Two-points boundary value problem (12)-(13) for a system of N ordinary differential equations with respect to the variable coefficients $a_k(x)$, $k = \overline{1,N}$ of the Kantorovich method (for the N-order approximation) considering the trial functions of the form (9);

3°. For the first order Kantorovich approximation on a domain with two plane solid walls, the Ritz variational method is applied and the Ritz algebraic system is constructed by choosing trigonometric trial functions (in the n-order Ritz approximation);

4°. In the numerical application we effectively found the Ritz solution (20), the approximate stream function (21) and the approximate equation of the stream lines (22), in order to test the exactity of the Kantorovich-Ritz approximate method (Table 1).

REFERENCES

1. N.M Beliaev and A.A. Riadno, *Methods of Theory of Heat Conduction*, 1, Moscow, Vyshaia Skola,1982 (in Russian).
2. P. Brădeanu, *Mecanica fluidelor*, Ed. Tehnică, Bucureşti, 1973.
3. L.E. Elsgolts, *Differential Equations and the Calculus of Variations*, Mir Publishers, Moscow, English transl., 1977.
4. L.V. Kantorovich and V.I. Krylov, *Approximate Methods of Higher Analysis*, Fizmatgiz, 1962 (in Russian).
5. S.G. Mikhlin, *Variational Methods in Mathematical Physics*, Nauka, Moscow, 1970 (in Russian).
6. S.G. Mikhlin, *Ecuaţii cu derivate parţiale*, Ed. Ştiinţifică şi Enciclopedică, Bucureşti,1973.

Received 18.03.1996

Romanian Academy
"T. Popoviciu" Institute of Numerical Analysis
P.O. Box 68, 3400 Cluj-Napoca
Romania