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ON THE APPROXIMATION BY FAVARD-SZASZ TYPE OPERATORS

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In 1969, A. Jakimovski and D. Leviatan [4] introduced a Favard-Szasz type operator, by means of Appell polynomials. One considers $g(z)=\sum_{n=0}^{\infty}a_nz^n$ an analytic function in the disk $|z|<R,\ R>1$, where $g(1)\neq 0$. It is known that the Appell polynomials $p_k(x),\ k\geq 0$ can be defined by

$$g(u)e^{ux}=\sum_{k=0}^{\infty}p_k(x)u^k \tag{1}$$

To a function $f:[0,\infty)\to R$ one associates the Jakimovski-Leviatan operator

$$(P_nf)(x)=\frac{e^{-nx}}{g(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right) \tag{2}$$

B. Wood [6] has proved that the operator P_n is positive in $[0,\infty)$ if and only if $\frac{a_n}{g(1)}\geq 0,\ n=0,1,\ldots$ The case $g(z)\equiv 1$ yields the classical operators of Favard-Szasz

$$(S_nf)(x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right)$$

In [4] A. Jakimovski and D. Leviatan have obtained several approximation properties of these operators. Let us mention some of these.

We will denote by E the class of functions of exponential type, which have the property that $|f(t)|\leq e^{At}$, *for each* $t\geq 0$ and some finite number A. Their basic theorem can be stated as follows: *If* $f\in C[0,\infty)\cap E$ *then* $\lim_{n\to\infty}(P_nf)(x)=f(x)$, *the convergence being uniform in each compact* $[0,a]$.

The aim of this paper is to study the order of approximation of the function f by means of the linear positive operator, P_n. We need the values of the operator, P_n for the monomials e_0, e_1, e_2, where $e_i(t) = t^i$, $i \in \{0, 1, 2\}$.
In [2] we found:

$$(3)\qquad \begin{aligned} (P_n e_0)(x) &= x \\ (P_n e_1)(x) &= x + \frac{1}{n}\frac{g'(1)}{g(1)} \\ (P_n e_2)(x) &= x^2 + \frac{x}{n}\left(1 + 2\frac{g'(1)}{g(1)}\right) + \frac{1}{n^2}\frac{g''(1) + g'(1)}{g(1)} \end{aligned}$$

In order to establish the main results of this paper we need the following:

DEFINITION 1. For $t \geq 0$ the second modulus of continuity of $f \in C_B[0,\infty)$ is

$$\omega_2(f;t) = \sup_{h \leq t}\left\| f(\circ + 2h) - 2f(\circ + h) + f(\circ)\right\|_{C_B}$$

where $C_B[0,\infty)$ is the class of real valued functions defined on $[0,\infty)$ which are bounded and uniformly continuous with the norm $\|f\|_{C_B} = \sup\limits_{x \in [0,\infty)} |f(x)|$

DEFINITION 2. [1]. The Peetre K-functional of function $f \in C_B$ is defined by

$$K(f;t) = \inf_{g \in C_B^2}\left\{\|f - g\|_{C_B} + t\|g\|_{C_B^2}\right\}$$

where $C_B^2 = \{f \in C_B | f', f'' \in C_B\}$ with the norm $\|f\|_{C_B^2} = \|f\|_{C_B} + \|f'\|_{C_B} + \|f''\|_{C_B}$

It is known that

$$(4)\qquad K(f;t) \leq A_1\left\{\omega_2\left(f;\sqrt{t}\right) + \min(1,t)\|f\|_{C_B}\right\}$$

for all $t \in [0,\infty)$. The constant A_1 is independent of t and f.

LEMMA 1. *If $z \in C^2[0,\infty)$ and (P_n) is a sequence of positive linear operators with the property $P_n e_0 = e_0$, then*

$$(5)\qquad |(P_n z)(x) - z(x)| \leq \|z'\|\sqrt{\left(P_n(t-x)^2\right)(x)} + \frac{1}{2}\|z''\|\left(P_n(t-x)^2\right)(x)$$

The proof is analogous to the proof of theorem 2 [3].

THEOREM 1. *If $f \in C[0,a]$, then for any $x \in [0, a]$, we have*

$$|(P_n f)(x) - f(x)| \leq \frac{2h}{a}\|f\| + \frac{3}{4}\left(3 + \frac{a}{h}\right)\omega_2(f;h),$$

where $h = \sqrt{\dfrac{x}{n} + \dfrac{1}{n^2}\dfrac{g''(1) + g'(1)}{g(1)}}$

Proof: Let f_h be the Steklov function attached to the function f. We will use a result given by V. V. Juk [5]: if $f \in C[a,b]$ and $h \in \left(0, \frac{b-a}{2}\right)$, then $\|f - f_h\| \leq \frac{3}{4}\omega_2(f;h)$ and $\|f_h''\| \leq \frac{3}{2}\frac{1}{h^2}\omega_2(f;h)$. Since $(P_n e_0)(x) = e_0$, we can write

$$\begin{aligned} |(P_n f)(x) - f(x)| &\leq |(P_n(f - f_h))(x)| + |(P_n f_h)(x) - f_h(x)| + |f_h(x) - f(x)| \leq \\ &\leq 2\|f - f_h\| + |(P_n f_h)(x) - f_h(x)| \end{aligned}$$

Using relation (5) for the function $f_h \in C^2[0,a]$, it results:

$$|(P_n f_h)(x) - f_h(x)| \leq \|f_h'\|\sqrt{\left(P_n(t-x)^2\right)(x)} + \frac{1}{2}\|f_h''\|\left(P_n(t-x)^2\right)(x)$$

In accordind with a results from [3] and [5], we obtain:

$$\|f_h'\| \leq \frac{2}{a}\|f_h\| + \frac{a}{2}\|f_h''\| \leq \frac{2}{a}\|f\| + \frac{a}{2}\|f_h''\| \leq \frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h)$$

and so it results that

$$\begin{aligned} |(P_n f_h)(x) - f_h(x)| &\leq \left(\frac{2}{a}\|f\| + \frac{3a}{4}\frac{1}{h^2}\omega_2(f;h)\right)\sqrt{\left(P_n(t-x)^2\right)(x)} + \\ &+ \frac{3}{4}\frac{1}{h^2}\omega_2(f;h)\left(P_n(t-x)^2\right)(x) \end{aligned}$$

By inserting into it $h = \sqrt{\left(P_n(t-x)^2\right)(x)} = \sqrt{\dfrac{x}{n} + \dfrac{1}{n^2}\dfrac{g''(1) + g'(1)}{g(1)}}$ we obtain

$$|(P_n f_h)(x) - f_h(x)| \leq \frac{2}{a}\|f\|h + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h)$$

Now we can write that

$$\begin{aligned} |(P_n f)(x) - f(x)| &\leq \frac{3}{2}\omega_2(f;h) + \frac{2h}{a}\|f\| + \frac{3a}{4}\frac{1}{h}\omega_2(f;h) + \frac{3}{4}\omega_2(f;h) = \\ &= \frac{2h}{a}\|f\| + \frac{3}{4}\omega_2(f;h)\left(3 + \frac{a}{h}\right) \end{aligned}$$

and so the theorem is proved.

THEOREM 2. *For every function* $f \in C_B^2[0,\infty)$ *we have*

$$\left|(P_n f)(x) - f(x)\right| \le \frac{1}{n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\|f\|_{C_B^2},\quad x\in[0,\infty)$$

Proof. Applying the Taylor expansion to $f \in C_B^2$, we have

$$(P_n f)(x) - f(x) = f'(x)(P_n(t-x))(x) + \frac{1}{2}f''(\xi)\left(P_n(t-x)^2\right)(x) \text{ where } \xi\in(t,x).$$

By making use of (3), we obtain

$$\left|(P_n f)(x) - f(x)\right| \le \|f'\|_{C_B}\frac{1}{n}\frac{g'(1)}{g(1)} + \frac{1}{2}\|f''\|_{C_B}\left(\frac{x}{n} + \frac{1}{n^2}\frac{g''(1)+g'(1)}{g(1)}\right) =$$

$$= \|f'\|_{C_B}\frac{1}{n}\frac{g'(1)}{g(1)} + \frac{1}{2}\|f''\|_{C_B}\frac{1}{n}\left(x + \frac{1}{n}\frac{g''(1)+g'(1)}{g(1)}\right) \le$$

$$\le \|f'\|_{C_B}\frac{1}{n}\frac{g'(1)}{g(1)} + \|f''\|_{C_B}\frac{1}{n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right) \le$$

$$\le \frac{1}{n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\left(\|f'\|_{C_B} + \|f''\|_{C_B}\right) \le \frac{1}{n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\|f\|_{C_B^2}$$

THEOREM 3. *For* $f \in C_B[0,\infty)$, *we have*

$$\left|(P_n f)(x) - f(x)\right| \le 2A_1\left\{\omega_2(f;h) + \lambda_n(x)\|f\|_{C_B}\right\}$$

where $h = \sqrt{\frac{1}{2n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)}$, A_1 *being a constant independent of t and f and*

$$\lambda_n(x) = \begin{cases} \frac{1}{2n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right) & ,\text{ if } x + \frac{g''(1)+g'(1)}{g(1)} < 2n \\ 1 & ,\text{ if } x + \frac{g''(1)+g'(1)}{g(1)} \ge 2n \end{cases}$$

Proof. We will use theorem 2, the Peetre K-functional and relation (4).

For $f \in C_B$ and $z \in C_B^2$, we can write

$$\left|(P_n f)(x) - f(x)\right| \le \left|(P_n f)(x) - (P_n z)(x)\right| + \left|(P_n z)(x) - z(x)\right| + \left|z(x) - f(x)\right| \le$$

$$\le 2\|f-z\|_{C_B} + \frac{1}{n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\|z\|_{C_B^2} = 2\left(\|f-z\|_{C_B} + \frac{1}{2n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\|z\|_{C_B^2}\right)$$

Because the left side of this inequality does not depend on the function $z \in C_B^2$, it results that:

$$\left|(P_n f)(x) - f(x)\right| \le 2K\left(f;\frac{1}{2n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\right) \le$$

$$\le 2A_1\left\{\omega_2\left(f;\sqrt{\frac{1}{2n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)}\right) + \min\left(1,\frac{1}{2n}\left(x + \frac{g''(1)+g'(1)}{g(1)}\right)\right)\|f\|_{C_B}\right\}.$$

This completes the proof of this theorem.

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