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EXTENSION OF BILINEAR OPERATORS AND BEST APPROXIMATION IN 2-NORMED SPACES

S. COBZAŞ and C. MUSTĂŢA
(Cluj-Napoca)

1. INTRODUCTION

In 1965 S. Gähler [6] defined 2-normed spaces and studied their basic properties. Since then the field has considerably grown, the research being directed to obtain a theory similar to that of normed spaces. A key result in developing such a theory is a Hahn-Banach type theorem for bilinear functional on 2-normed spaces. But, as remarked S. Gähler [7, p.345], a general Hahn-Banach theorem doesn't hold in this setting. Some extension theorems for bounded bilinear functionals defined on subspaces of the form $Z \times [b]$ to $X \times [b]$ (X–a 2-normed space, Z a subspace of X and $[b]$ - the subspace of X spanned by $b \in X \setminus \{0\}$) were proved by A. G. White [18], S. Mabizela [13] and I. Franić [4]. In [2] it was shown that all these results follow directly from the classical Hahn-Banach theorem for linear functionals on seminormed spaces.

Inspired by some results of L. Nachbin [14], [15] and of J. Lindenstrauss [12], I. Beg and M. Iqbal [1] proved some extension theorems for bounded or compact bilinear operators defined also on subspaces of the form $Z \times [b]$. In this paper we shall show that, again, all these results follow directly from the corresponding results for linear operators on normed spaces. The key tool will be a result relating the bilinear operators from $Z \times [b]$ to a semi-normed space (Y, q) and the linear operators from Z to Y (Proposition 3.2 below). The extension results are applied to obtain some duality results for best approximation in spaces of bounded bilinear operators.

2. BOUNDED LINEAR OPERATORS ON SEMINORMED SPACES

Let (X, p) and (Y, q) be two seminormed spaces. It is well known that a linear operator $A : X \to Y$ is continuous if and only if it is *bounded* (or *Lipschitz*), i.e. there exists a number $L \geq 0$ such that

$$q(Ax) \le L \cdot p(x), \quad \text{for all } x \in X. \tag{2.1}$$

A number $L \ge 0$ verifying (2.1) is called a *Lipschitz constant* for A. For a bounded linear operator $A : X \to Y$ define by

$$\|A\| = \sup\{q(Ax) : x \in X,\ p(x) \le 1\}, \tag{2.2}$$

the *norm* of the operator A. The following results are well known in the case of normed spaces (see e.g. [3]). Since their proofs can be transposed with slight and obvious modifications to the case of seminormed spaces we shall omit them. Denote by $L(X, Y)$ the space of all bounded linear operators from X to Y.

PROPOSITION 2.1 *Let (X, p) and (Y, q) be seminormed spaces. Then the following assertions hold:*

1° *If $A \in L(X, Y)$ then the number $\|A\|$, defined by (2.2) is the smallest Lipschitz constant for A, i.e.*

$$\|A\| = \min\{L \ge 0 : L \text{ is a Lipschitz constant for } A\}$$

2° *The application $\|\cdot\| : L(X,Y) \to [0,\infty)$ is a seminorm on $L(X, Y)$ which is a norm if and only if q is a norm on Y.*

3° *The seminormed space $(L(X,Y), \|\cdot\|)$ is complete if (and only if when $q \ne 0$) the seminormed space (Y, q) is complete.*

3. BOUNDED BILINEAR OPERATORS ON 2-NORMED SPACES

Let X be a real vector space of dimension at least 2. An application $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ is called a *2-norm* on X if

BN 1) $\|x, y\| = 0$ if and only if x, y are linearly dependent,

BN 2) $\|x, y\| = \|y, x\|$,

BN 3) $\|\lambda x, y\| = |\lambda|\,\|x, y\|$,

BN 4) $\|x + y, z\| \le \|x, z\| + \|y, x\|$,

for all $x, y, z \in X$ and all $\lambda \in \mathbf{R}$ (see [6]).

A 2-normed space is a real linear space equipped with a 2-norm $\|\cdot,\cdot\|$. If $(X, \|\cdot,\cdot\|)$ is a 2-normed space and $b \in X$ then the functional $p_b : X \to [0, \infty)$,

defined by $p_b(x) = \|x, b\|, x \in X$, is a seminorm on X. The locally convex topology generated by the family $P = \{p_b : b \in X\}$ of seminorms is called the *natural topology* of X induced by the 2-norm $\|\cdot,\cdot\|$, (see [6]).

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and X_1, X_2 subspaces of X. A *bilinear operator* is an application T from $X_1 \times X_2$ to a seminormed space (Y, q) such that:

(BL) $T(\cdot, y) : X_1 \to Y$ and $T(x, \cdot) : X_2 \to Y$ are linear operators, for all $x \in X_1$ and all $y \in X_2$.

A bilinear operator $T : X_1 \times X_2 \to Y$ is called *bounded* if there exists a number $L \ge 0$ such that

$$q(T(x, y)) \le L\|x, y\|, \quad \text{for all } (x, y) \in X_1 \times X_2. \tag{3.1}$$

A number $L \ge 0$ verifying (3.1) is called a Lipschitz constant for T. A *bilinear functional* is a bilinear operator $F : X_1 \times X_2 \to \mathbf{R}$. As it was shown by A. G. White [18] in the case of bilinear functionals, and by I. Beg and M. Iqbal [1] in general, the boundedness of a bilinear operator can be characterized by a kind of a continuity condition, called 2-continuity by S. Gähler [7]. It turns that the 2-continuity of a bilinear operator at (0, 0) implies its 2-continuity on the whole $X_1 \times X_2$. A typical example of a (nonlinear) functional which is continuous on $X \times X$ is the 2-norm $\|\cdot,\cdot\|$. As remarked S. Gähler [7] this notion of 2-continuity is different from the continuity with respect to the natural product topology on $X \times X$.

For a bounded bilinear operator $T : X_1 \times X_2 \to Y$ define

$$v(T) = \sup\{q(T(x, y)) : (x, y) \in X_1 \times X_2,\ \|x, y\| \le 1\} \tag{3.2}$$

and denote by $L_2(X_1 \times X_2, Y)$ the linear space of all bounded bilinear operators from $X_1 \times X_2$ to Y. As in the case of linear operators on normed spaces one can easily prove:

PROPOSITION 3.1 *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space X_1, X_2 subspaces of X and (Y, q) a seminormed space. Then the following assertions hold:*

1° *If $T \in L_2(X_1 \times X_2, Y)$ then $v(T)$ is the smallest Lipschitz constant for T, i. e.*

$$v(T) = \min\{L \ge 0 : L \text{ is a Lipschitz constant for } T\}. \tag{3.3}$$

2° *The application $v : L_2(X_1 \times X_2, Y) \to [0, \infty)$ is a seminorm on $L_2(X_1 \times X_2, Y)$ which is a norm if and only if q is a norm on Y.*

3° *The seminormed space $(L_2(X_1 \times X_2, Y), v)$ is complete if the seminormed space (Y, q) is complete.*

Remark. The completeness of $L_2(X \times X, Y)$ for the case of a Banach space (Y, q) was proved by A. G. White Jr. [18].

For an element $b \in X \setminus \{0\}$ denote by $[b]$ the subspace of X spanned by b (i.e. $[b] = \mathbf{R} \cdot b$). If Z is a subspace of X let p_b denote the seminorm $p_b(z) = \|z, b\|$, $z \in Z$. The bilinear operators from $Z \times [b]$ to a seminormed space (Y, q) and the linear operators between the seminormed spaces (Z, p_b) and (Y, q) are related as in the following proposition. Here $\|A\|$ and $\nu(T)$ denote the norms of a linear operator A (cf. (2.2)) and respectively of a bilinear operator T (cf. (3.2)).

PROPOSITION 3.2 1° *If $T : Z \times [b] \to Y$ is a bounded bilinear operator then the operator $A : (Z, p_b) \to (Y, q)$ defined by $Az = T(z,b)$, $z \in Z$, is a continuous linear operator and*

$$\|A\| = \nu(T). \tag{3.4}$$

2° *Conversely, if $A : (Z, p_b) \to (Y, q)$ is a continuous linear operator, then the operator $T : Z \times [b] \to Y$, defined by $T(z, \alpha b) = \alpha \cdot Az$, for $z \in Z$ and $\alpha \in \mathbf{R}$, is a bounded bilinear operator and*

$$\nu(T) = \|A\| \tag{3.5}$$

Proof. 1° If $T : Z \times [b] \to Y$ is a bilinear operator, it is immediate that the operator $A : Z \to Y$ defined by $Az = T(z, b)$, $z \in Z$, is linear. Since

$$q(Az) = q(T(z,b)) \le \nu(T)\|z,b\| = \nu(T) \cdot p_b(z),$$

for all $z \in Z$, it follows that A is continuous and $\|A\| \le \nu(T)$.

But

$$q(T(z,\alpha b)) = q(T(\alpha z,b)) = q(A(\alpha z)) \le \|A\| \cdot p_b(\alpha z) =$$

$$= \|A\| \cdot \|\alpha z,b\| = \|A\| \cdot \|z,\alpha b\|,$$

for all $z \in Z$ and all $\alpha \in \mathbf{R}$ implying $\nu(T) \le \|A\|$ and $\|A\| = \nu(T)$.

2° Suppose now that $A : (Z, p_b) \to (Y, q)$ is a continuous linear operator and let $T : Z \times [b] \to Y$ be defined by $T(z, \alpha b) = \alpha \cdot Az$, for $z \in Z$ and $\alpha \in \mathbf{R}$. It is obvious that T is a bilinear operator and from

$$q(T(z,\alpha b)) = q(\alpha \cdot Az) = q(A(\alpha z)) \le \|A\| \cdot p_b(\alpha z) = \|A\| \cdot \|\alpha z,b\| = \|A\| \cdot \|z,\alpha b\|$$

we get $\nu(T) \le \|A\|$.

The equalities $A(\alpha z) = T(z, \alpha b)$, $\|\alpha z, b\| = \|z, \alpha b\|$, and the definitions of the norms $\|A\|$ and $\nu(T)$ (relations (2.2) and (3.2) respectively) imply

$$\|A\| = \sup\{q(Az) : z \in Z,\ p_b(z) \le 1\} = \sup\{q(Az) : z \in Z, \|z, b\| \le 1\}$$

$$\le \sup\{q(A(\alpha z)) : z \in Z, \alpha \in \mathbf{R},\ \|\alpha z, b\| \le 1\}$$

$$= \sup\{q(T(z,\alpha b)) : z \in Z, \alpha \in \mathbf{R},\ \|z, \alpha b\| \le 1\} = \nu(T),$$

showing that $\nu(T) = \|A\|$ □

4. NORM PRESERVING EXTENSIONS OF BILINEAR OPERATORS

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space and X_1, X_2 linear subspaces of X. A *norm-preserving extension* of a bounded bilinear operator T from $X_1 \times X_2$ to a seminormed space (Y, q) is a bounded bilinear operator $\widetilde{T} : \widetilde{X}_1 \times \widetilde{X}_2 \to Y$, (where $\widetilde{X}_1$ and $\widetilde{X}_2$ are linear subspaces of X containing X_1 respectively X_2), such that

i) $\widetilde{T}(x_1, x_2) = T(x_1, x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$ and

ii) $\nu(\widetilde{T}) = \nu(T)$.

For two seminormed spaces (X,p) and (Y,q) a *norm-preserving extension* of a bounded linear operator A, defined on a subspace Z of X and taking values in Y, is a bounded linear operator $\widetilde{A}$ defined on a subspace $\widetilde{Z}$ of X, $Z \subseteq \widetilde{Z}$, and taking values in Y, such that

i) $\widetilde{A}z = Az$, for all $z \in Z$, and

ii) $\|\widetilde{A}\| = \|A\|$.

A normed space (Y, q) is said to have the *extension property* if for any normed space (X, p), every continuous linear operator A, defined on a subspace Z of X and taking values in Y, has a norm-preserving extension $\widetilde{A} : X \to Y$. A normed space (Y, q) is said to have the *binary intersection property* if every family of mutually intersecting closed convex balls in Y has a nonvoid intersection. By a famous result of L. Nachbin [14] (see also [15]) a normed space (Y, q) has the extension property if and only if it has the binary intersection property. The binary intersection property, and the extension property can be defined in a similar way for seminormed spaces, yielding a seminormed version of Nachbin's result.

I. Beg and M. Iqbal [1] transposed the sufficiency part of Nachbin's theorem to bilinear operators defined on subspaces of the form $Z \times [b]$. We shall show that this result is an immediate consequence of Nachbin's result and of Proposition 3.2.

First we prove the following result:

PROPOSITION 4.1 *Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space (Y, q) a seminormed space. Let Z and $\widetilde{Z}$ be subspaces of X such that $Z \subseteq \widetilde{Z}$ and let $b \in X \setminus \{0\}$. Suppose that the operators $T : Z \times [b] \to Y$, $\widetilde{T} : \widetilde{Z} \times [b] \to Y$ and $A : Z \to Y$ respectively $\widetilde{A} : \widetilde{Z} \to Y$, are related as in Proposition 3.2.*

Then $\widetilde{T}$ is a norm-preserving extension of T if and only if the corresponding linear operator $\widetilde{A}$ is a norm-preserving extension of A.

Proof. The proof is an immediate consequence of the relations $T(z, \alpha b) = \alpha Az$, $(z,\alpha) \in Z\times\mathbf{R}$, $\nu(T) = \|A\|$ and $\widetilde{T}(\widetilde{z},\alpha b) = \alpha \cdot \widetilde{A}\widetilde{z}$, $(\widetilde{z},\alpha) \in \widetilde{Z} \times \mathbf{R}$, $\nu(\widetilde{T}) = \|\widetilde{A}\|$, relating the operators T and A, respectively $\widetilde{T}$ and $\widetilde{A}$ □

Let us agree to say that a seminormed space (Y, q) has the *restricted extension property for bilinear operators* if for any 2-normed space $(X,\|\cdot,\cdot\|)$ every bounded bilinear operator $T: Z \times [b] \to Y$ (Z a subspace of X and $b \in X\setminus\{0\}$ has a norm-preserving extension $\widetilde{T}: X \times [b] \to Y$. Using these terms, Proposition 4.1 can be restated as follows:

COROLLARY 4.2 *A seminormed space (Y, q) has the restricted extension property for bilinear operators if it has the extension property for linear operators.*

Observing that the proof of sufficiency part of Nachbin's theorem [14, p.31] remains valid when all spaces are supposed to be seminormed we get:

COROLLARY 4.3 ([*1, Th. 2.1*]). *If a seminormed space (Y, q) has the binary intersection property, then it has the restricted extension property for bilinear operators.*

Remark. We do not know whether the necessity part of Nachbin's theorem remains valid for bilinear operators too: Must a seminormed space (Y, q) verifying the restricted extension property for bilinear operators have the binary intersection property ?

The extension result for operators defined on condimension one subspaces, proved by J. Lindenstrauss [12, Lemma 5.2], can be transposed to bilinear operators too.

PROPOSITION 4.4 *Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space, Z a codimension one subspace of X and $b \in X\setminus\{0\}$. A bounded bilinear operator T from $Z \times [b]$ to a seminormed space (Y, q) admits a norm-preserving extension $\widetilde{T}: X \times [b] \to Y$ if and only if there exists $u \in X\setminus Z$ such that*

$$\bigcap\{B_q(T(z,b), \nu(T)\,\|u - z, b\|: z \in Z)\} \neq \varnothing. \tag{4.1}$$

Proof. If $u \in X\setminus Z$ then $X = Z + \mathbf{R}u$ and any bilinear extension $\widetilde{T}: X \times [b] \to Y$ of T is completely determined through the formula

$$\widetilde{T}(z + \alpha u, \beta b) = T(z, \beta b) + \alpha\beta y_0, \tag{4.2}$$

by its value y_0 at (u, b). Consequently, $\widetilde{T}$ is a norm-preserving extension of T if and only if

$$q(\widetilde{T}(z + \alpha u, \beta b) \le \nu(T) \cdot \|z + \alpha u, \beta b\|), \tag{4.3}$$

for all $z \in Z$ and all $\alpha,\beta \in \mathbf{R}$. Supposing $\alpha \cdot \beta \neq 0$ and deleting by $|\alpha \cdot \beta| > 0$, one obtains successively:

$$q(\widetilde{T}(z + \alpha u, \beta b)) \le \nu(T) \cdot \|z + \alpha u, \beta b\| \iff$$

$$q\left(\widetilde{T}\left(\alpha^{-1}z + u, b\right)\right) \le \nu(T) \cdot \left\|\alpha^{-1}z + u, b\right\| \iff$$

$$q(y_0 - T(z', b)) \le \nu(T) \cdot \|u - z', b\|,$$

for all $z' = -\alpha^{-1}z \in Z$. This last relation is equivalent to:

$$y_0 \in \bigcap\{B_q(T(z,b), \nu(T) \cdot \|u - z, b\|): z \in Z\}.$$

Since, for $\alpha = 0$ $q(\widetilde{T}(z, \beta b)) = q(T(z,\beta b)) \le \nu(T) \cdot \|z, \beta b\|$ and for $\beta = 0$ $\widetilde{T}(z + \alpha u, 0) = 0$, the proposition is proved □.

Remark. 1° From the proof it is clear that if relation (4.3) holds for an element $u_0 \in X\setminus Z$, then it holds for any other element $u \in X\setminus Z$.

2° Proposition 4.4 appears in [1, Proposition 3.3] in a slightly different form.

5. COMPACT BILINEAR OPERATORS

The aim of this section is to show how some extension results for compact operators on normed spaces, proved by J. Lindestrauss [12], can be transposed to bilinear operators on 2-normed spaces. The basic tool used in doing this will be again Proposition 3.2.

Roughly speaking, a compact bilinear operator is a bilinear operator mapping bounded sets into relatively compact ones. We shall consider three boundedness notions in 2-normed spaces and three corresponding compactness notions for bilinear operators.

Let $(X,\|\cdot,\cdot\|)$ be a 2-normed space and $b \in X\setminus\{0\}$. A subset V of X is called p_b*-bounded* if $\sup p_b(V) < \infty$. The set V is called *bounded* if it is p_b-bounded for all $b \in X\setminus\{0\}$ (and obviously for all $b \in X$). This is nothing else than the boundedness of V with respect to the natural locally convex topology of X induced by the 2-norm $\|\cdot,\cdot\|$. Finally, we call a subset W of $X\times X$ *2-norm bounded* provided $\sup\{\|x, y\|: (x, y) \in W\} < \infty$. The corresponding boundedness notions for sequences in X or in $X \times X$ are defined in an obvious way.

Let X_1, X_2 be linear subspaces of a 2-normed space $(X,\|\cdot,\cdot\|)$ and let (Y, q) be a seminormed space. A bilinear operator $T: X_1 \times X_2 \to Y$, is called *separately compact* (*s-compact* for short) if $\{T(x_n, y_n)\}$ contains a convergent subsequence for every bounded sequence $\{x_n\}$ in X_1 and every bounded sequence $\{y_n\}$ in X_2. The operator T is called compact if $\{T(x_n, y_n)\}$, contains a convergent subsequence for every 2-norm bounded sequence $\{(x_n, y_n)\}$ in $X_1 \times X_2$.

We have:

PROPOSITION 5.1 *Every compact operator is bounded.*

Proof. Let $T: X_1 \times X_2 \to Y$ be a compact bilinear operator. Supposing T not bounded then, by (3.2), we can choose a sequence $\{(x_n, y_n)\}$ in $X_1 \times X_2$ such that $\|x_n, y_n\| \le 1$ and $q(T(x_n, y_n)) > n$, for all $n \in \mathbf{N}$. It follows that the sequence $\{T(x_n, y_n)\}$ has no convergent subsequences □

Consider now the case of a bilinear operator $T: Z \times [b] \to Y$, where Z is a subspace of the 2-normed space $(X,\|\cdot,\cdot\|)$ and $b \in X \setminus \{0\}$. We call the operator T *p_b-compact* if $\{T(z_n, \alpha_n b)\}$ contains a convergent subsequence for every p_b-bounded sequence $\{z_n\}$ in Z and every bounded sequence $\{\alpha_n\}$ in $\mathbf{R}$. In this case these three notions of compactness are related as follows:

PROPOSITION 5.2 *Let $T: Z \times [b] \to Y$ be a bilinear operator. Then*

$$T \text{ compact} \Rightarrow T p_b - \text{compact} \Rightarrow Ts - \text{compact}.$$

Proof. T compact $\Rightarrow Tp_b$– compact

If $\{z_n\}$ is a p_b-bounded sequence in Z and $\{\alpha_n\}$ is a bounded sequence of real numbers, then the equality $\|z_n, \alpha_n b\| = |\alpha_n| \cdot \|z_n, b\|$ implies $\sup_n \|z_n, \alpha_n b\| < \infty$. The operator T being compact it follows that $\{T(z_n, \alpha_n b)\}$ contains a convergent subsequence.

To prove the second implication we need a lemma:

LEMMA 5.3 *Let $b \in X \setminus \{0\}$ and $\{\alpha_n\} \subset \mathbf{R}$. The sequence $\{\alpha_n b\}$ is bounded in X if and only if the sequence $\{\alpha_n\}$ is bounded in $\mathbf{R}$.*

Proof. By definition, a 2-normed space has dimension at least 2, so that there exist $a \in X$ with $\|a, b\| > 0$ (Axiom BN 1). The boundedness of $\{\|\alpha_n b, a\|\}$, and the equality $\|\alpha_n b, a\| = |\alpha_n| \cdot \|a, b\|$ imply the boundedness of the sequence $\{\alpha_n\}$.

Conversely, if $\{\alpha_n\}$ is a bounded sequence of real numbers then the equality $\|\alpha_n b, c\| = |\alpha_n| \cdot \|b, c\|$ implies that the sequence $\{\|\alpha_n b, c\|\}$ is bounded for every $c \in X$. Lemma is proved.

Prove now that:

$$T p_b - \text{compact} \Rightarrow Ts - \text{compact}$$

Let $\{z_n\} \subseteq Z$ and $\{\alpha_n b\}$ be bounded sequences. It follows that the sequence $\{z_n\}$ is p_b-bounded and, by Lemma 5.3, the sequence $\{\alpha_n\}$ is bounded too. Since T is p_b-compact, the sequence $\{T(z_n, \alpha_n b)\}$ will contain a convergent subsequence, proving that T is a p_b-compact bilinear operator □.

Concerning the compactness properties of a bilinear operator $T: Z \times [b] \to Y$ and of the associated operator $A: Z \to Y$ (in the sense of Proposition 3.2) one can prove:

PROPOSITION 5.4 *A bilinear operator $T: Z \times [b] \to Y$ is p_b-compact if and only if the associated linear operator $A: (Z, p_b) \to (Y, q)$ is compact.*

Proof. Suppose that the bilinear operator $T: Z \times [b] \to Y$ is p_b-compact and let $\{z_n\}$ be a bounded sequence in the seminormed space (Z, p_b). It follows that $\{z_n\}$ is a p_b-bounded sequence in Z and, consequently, $\{T(z_n, b)\}$ will contain a convergent subsequence $\{T(z_{n_k}, b)\}$. Since $\left\{T\left(z_{n_k}, b\right)\right\} = Az_{n_k}$ it follows that $\left\{Az_{n_k}\right\}$ is a convergent subsequence of $\left\{Az_{n_k}\right\}$, showing that the operator A is compact.

Conversely, let $A: (Z, p_b) \to (Y, q)$ be a compact linear operator. If $\{z_n\}$ is a p_b-bounded sequence in Z and $\{\alpha_n\}$ is a bounded sequence in $\mathbf{R}$, then $\{Az_n\}$ contains a convergent subsequence $\left\{Az_{n_k}\right\}$. Taking a convergent subsequence $\left\{\alpha_{n_{k_j}}\right\}$ of $\left\{\alpha_{n_k}\right\}$ it follows that $T\left(z_{n_{k_j}}, \alpha_{n_{k_j}} b\right) = \alpha_{n_{k_j}} \cdot Az_{n_{k_j}}$, $j \in \mathbf{N}$, is a convergent subsequence of $\{T(z_n, \alpha_n b)\}$. Therefore the operator T is p_b-compact □.

The following result was proved by J. Lindestrauss [12, Th. 5.4], in the case of linear operators on normed spaces and by I. Beg and M. Iqbal [1, Th. 3.5] in the case of bilinear operators on 2-normed spaces. A normed space (Y, q) is said to have the *finite 2-intersection property* (F.2.I.P.) if any finite collection of mutually intersecting closed balls in Y has nonvoid intersection.

PROPOSITION 5.5 *Let (Y, q) be a Banach space with the (F.2.I.P.), $(X,\|\cdot,\cdot\|)$ a 2-normed space, Z a codimension one subspace of X, $b \in X \setminus \{0\}$ and $T: Z \times [b] \to Y$ a p_b-compact bilinear operator. Then, for every $\varepsilon > 0$ there exists an extension $\tilde{T}: X \times [b] \to Y$ of T verifying $\nu(\tilde{T}) \le (1+\varepsilon) \cdot \nu(T)$.*

Proof. Let $A: (Z, p_b) \to (Y, q)$ be the linear operator associated to T according to Proposition 3.2. By Proposition 5.4, the operator A is compact and, by J. Lindenstrauss [12, Th. 5.4], there exists an extension $\tilde{A}: (X, p_b) \to (Y, q)$ of A, verifying $\|\tilde{A}\| \le (1+\varepsilon) \cdot \|A\|$. Appealing again to Proposition 3.2 it follows that the desired extension of T is given by $\tilde{T}(x, \alpha b) = \alpha \cdot \tilde{A}x$, $x \in X$, $\alpha \in \mathbf{R}$ □.

Remark. 1° I. Beg and M. Iqbal [1, Th. 3.5] proved Proposition 5.5 for compact bilinear operators following the ideas of the proof given by J. Lindenstrauss [12] for compact linear operators. By Proposition 5.2, a compact bilinear operator $T : Z \times \times [b] \to Y$ is p_b-compact, so that the result of I. Beg and M. Iqbal follows from Proposition 5.5.

2° We have used a seminormed version of Lindenstrauss' result which can be proved in the same way as in the case of normed spaces (The space Y could be also supposed to be only seminormed too).

6. UNIQUE EXTENSION AND UNIQUE BEST APPROXIMATION

The aim of this section is to prove some duality results relating the extension properties for bilinear operators and best approximation in spaces of bilinear operators. In the case of linear functionals on normed spaces the problem was first studied by R. R. Phelps [16]. For other related results see I. Singer's book [17].

Recall that, for a 2-normed space $(X,\|\cdot,\cdot\|)$, a normed space (Y,q) and two subspaces X_1, X_2 of X, we denote by $L_2(W, Y)$ the normed space of all bounded bilinear operators from $W = X_1 \times X_2$ to Y. If $\widetilde{X}_1 \supset X_1$ and $\widetilde{X}_2 \supset X_2$ are other two subspaces of X then the normed space $L_2(\widetilde{W}, Y)$ and $\widetilde{W}$ are defined similarly. The norms in $L_2(W, Y)$ and $L_2(\widetilde{W}, Y)$ will be denoted by the same symbol ν (see (3.2) and Proposition 3.1). For $T \in L_2(W, Y)$ denote by $\mathscr{E}(T)$ the set of all norm preserving extensions of T to $\widetilde{W}$, i.e.

$$\mathscr{E}(T) = \{\widetilde{T} \in L_2(\widetilde{W}, Y) : \widetilde{T}|_W = T \ \text{ and } \ \nu(\widetilde{T}) = \nu(T)\}. \tag{6.1}$$

The annihilator of W is $L_2(\widetilde{W}, Y)$ is defined by

$$W^{\perp} = \{S \in L_2(\widetilde{W}, Y) : S(W) = \{0\}\}. \tag{6.2}$$

As usual, for a nonvoid subset V of a normed space E and $x \in E$, denote by $d(x, V) = \inf\{\|x - v\| : v \in V\}$ the *distance* from x to V. An element $v_0 \in V$ satisfying $\|x - v_0\| = d(x, V)$ is called *a nearest point* to x in V (or a *best approximation element*). The set of nearest points to x in V is denoted by $P_V(x)$ and the set-valued operator $P_V : E \to 2^V$ is called the *metric projection operator* of E onto V. The set V is called *proximinal* if $P_V(x) \neq \varnothing$ and *Chebyshevian* if $P_V(x)$ is a singleton, for all $x \in E$.

We say that W has the *extension property with respect* to $\widetilde{W}$ if every bounded bilinear operator $T : W \to Y$ has a norm preserving extension $\widetilde{T} \in L_2(\widetilde{W}, Y)$. The following proposition shows that the extension properties of W and the best approximation properties of its annihilator are closely related.

PROPOSITION 6.1 *If the subspace W has the extension property with respect to $\widetilde{W}$, then its annihilator $W^{\perp}$ is a proximinal subspace of $L_2(\widetilde{W}, Y)$ and the following formulae hold*

$$d(S, W^{\perp}) = \nu(S|_W) \tag{6.3}$$

and

$$P_{W^{\perp}}(S) = S - \mathscr{E}(S|_W), \tag{6.4}$$

for any operator $S \in L_2(\widetilde{W}, Y)$.

Proof. If $S \in L_2(\widetilde{W}, Y)$ then $(S - T)|_W = S|_W$ and, by the definition of the norm ν (formula (3.2)), we have

$$\nu(S|_W) = \nu((S - T)|_W) \leq \nu(S - T),$$

for every $T \in W^{\perp}$, implying $\nu(S|_W) \leq d(S, W^{\perp})$. If $S \in L_2(\widetilde{W}, Y)$, is a norm-preserving extension of $S|_W$ then $T_0 = S - S_0 \in W^{\perp}$ and since $S_0 = S - T_0$ we can write

$$\nu(S|_W) = \nu(S_0) = \nu(S - T_0) \geq d(S, W^{\perp}),$$

showing that formula (6.3) holds and that $S - S_0$ is a nearest point to S in $W^{\perp}$ for any $S_0 \in \mathscr{E}(S|_W)$, i.e.

$$S - \mathscr{E}(S|_W) \subseteq P_{W^{\perp}}(S). \tag{6.5}$$

Suppose now that T_0 is a nearest point to S in $W^{\perp}$ and let $S_0 = S - T_0$. It follows $S_0|_W = S|_W$ and by (6.3)

$$\nu(S|_W) = d(S, W^{\perp}) = \nu(S - T_0) = \nu(S_0)$$

showing that S_0 is a norm-preserving extension of $S|_W$, i.e.

$$S - P_{W^{\perp}}(S) \subseteq \mathscr{E}(S|_W)$$

or equivalently

$$P_{W^{\perp}}(S) \subseteq S - \mathscr{E}(S|_W) \tag{6.6}$$

which together with relation (6.5) give (6.4).□

Let $(X, \|\cdot,\cdot\|)$ be a 2-normed space, Z a subspace of X and $b \in X \setminus \{0\}$. For a normed space (Y, q) let

$$Z_b^{\perp} = \{T \in L_2(X \times [b], Y): T(Z \times [b]) = \{0\}\}$$

denote the annihilator of $Z \times [b]$ in $L_2(X \times [b], Y)$. In this case Proposition 6.1 and Corollary 4.3 give:

COROLLARY 6.2 *Let (Y, q) be a normed space with the binary intersection property. Then $Z_b^{\perp}$ is a proximinal subspace of $L_2(X \times [b], Y)$ and the following formulae*

$$d(S, Z_b^{\perp}) = \nu\left(S|_{Z\times[b]}\right) \tag{6.7}$$

and

$$P_{Z_b^{\perp}}(S) = S - \mathscr{E}\left(S|_{Z\times[b]}\right) \tag{6.8}$$

hold for every $S \in L_2(X \times [b], Y)$.

Furthermore the annihilator $W^{\perp}$ is a Chebyshevian subspace of $L_2(X \times [b], Y)$ if and only if every $T \in L_2(Z \times [b], Y)$ has a unique norm preserving extension to $X \times [b]$.

REFERENCES

1. I. Beg, M. Iqbal, *Extension of linear 2-operators*, Mathematica Montesnigri **2** (1993), 1-10.
2. S. Cobzaş, C. Mustăţa, *Extension of bilinear functionals and best approximation in 2-normed spaces*, (to appear).
3. N. Dunford, J. T. Schwartz, *Linear Operators Part. I: General Theory*, Interscience Publishers, New York 1958.
4. I. Franić, *An extension theorem for bounded linear 2-functionals and applications*, Math. Japonica **40** (1994),79-85.
5. S. Gähler, *2-Metrische Räume und ihre Topologische Struktur*, Math. Nachr. **26** (1963/64), 115-148.
6. S. Gähler, *Linear 2-Normierte Räume*, Math. Nachr. **28** (1965), 1- 45.
7. S. Gähler, *Über 2-Banach-Räume*, Math. Nachr. **42** (1969), 335-347.
8. T. L. Hayden, *The extension of bilinear functionals*, Pacific. J. Math. **22** (1967), 99-108.
9. R. B. Holmes, *On the continuity of best approximation operators*, in Symposium on Infinite Dimensional Topology (R. D. Anderson Editor), Annals of Math. Studies, Princeton Univ. Press, Princeton 1972, 137-158.
10. R. B. Holmes, *Geometric Functional Analysis and its Application*, Springer-Verlag, Berlin 1975.
11. K. Iseki, *Mathematics on 2-normed spaces*, Bull. Korean. Math. Soc., **13** (1976), 127-136.
12. J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc. 48, (1964)
13. S. Mabizela, *On bounded 2-linear functionals*, Math. Japonica, **35** (1990), 51-55
14. L. Nachbin, *A theorem of Hahn-Banach type for linear transformation*, Trans. Amer. Math. Soc., **68** (1950), 28-46

15. L. Nachbin, *Some problems in extending and lifting continuous linear transformations*, in Proc. International Symposium on Linear Spaces, Jerusalem, 1960.
16. R. R. Phelps, *Uniqueness of Hahn-Banach extension and unique best approximation*, Trans Amer. Math. Soc., **95** (1960), 238-255.
17. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Publishing House of the Romanian Academy and Springer Verlag, Bucharest and Berlin,1970.
18. A. G. White Jr., *2-Banach spaces*, Math. Nachr. **42** (1969), 43-60.
19. A. G. White and Yeol Je Cho, *Linear mappings on linear 2-normed spaces*, Bull. Korean Math. Soc. **21** (1984),1-6.

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S. Cobzas
Faculty of Mathematics
Ro-3400 Cluj-Napoca
Romania

C. Mustăţa
"T. Popoviciu" Institute
of Numerical Analysis
O.P. 1 C.P. 68
Ro-3400 Cluj-Napoca
Romania