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POLYNOMIAL SPLINE COLLOCATION METHODS FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Consider the first-order Volterra integro-differential equation (VIDE):

$$(1.1) \qquad y'(t) = f(t, y(t)) + \int_0^t K(t, s, y(s))\mathrm{d}s, \quad t \in I := [0, T],$$

with initial condition $y(0) = y_0$. Here, the given functions $f : I \times R \to R$ and $K : S \times R \to R$ (with $S := \{(t, s) : 0 \le s \le t \le T\}$), are supposed to be sufficiently smooth for the initial-value problem for VIDE (1.1) to have a unique solution $y \in C^{\alpha}(I)$, with $\alpha \in N$ (see[3], [6]).

VIDE-s of the above form will be solved numerically in certain polynomial spline spaces. In order to describe these approximating spaces let $\Pi_N : 0 = t_0 < t_1 < \ldots < t_N = T$ (with $t_n = t_n^{(N)}$) be a mesh for the given interval I, and set

$$\sigma_0 := [t_0, t_1],\ \sigma_n := [t_n, t_{n+1}],\ h_n := t_{n+1} - t_n,\ n = 0, \ldots, N-1,$$
$$h = \max_{(n)}(t_{n+1} - t_n),$$
$$Z_N := \{t_n : n = 1, \ldots, N-1\},\ \overline{Z}_N = Z_N \cup \{T\}.$$

Moreover, let $\mathscr{P}_k$ denote the space of (real) polynomials of degree not exceeding k. We then define, for given integers m and d with $m \ge 1$ and $d \ge -1$,

$$S_{m+d}^{(d)}(Z_N) := \Big\{u : u(t)\big|_{t \in \sigma_n} =: u_n(t) \in \mathscr{P}_{m+d},\ n = 0, \ldots, N-1,$$
$$u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } j = 0, 1, \ldots, d \text{ and } t_n \in Z_N\Big\},$$

to be the space of polynomial splines of degree $m+d$ whose elements possess the knots Z_N and are d times continually differentiable on I. If $d=-1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinues at the knots Z_N.

In many papers, the problem of approximating the exact solution of initial-value problem for *VIDE* (1.1), has been solved by collocation method in polynomial splines spaces $S_m^{(0)}(Z_N)$ and $S_m^{(1)}(Z_N)$ (see [1], [2], [3]) or in polynomial splines space $S_{d+1}^{(d)}(Z_N)$ (see [6]). In this paper we shall construct an approximate solution in the space of polynomial spline functions $S_{m+d}^{(d)}(Z_N)$, with $m\geq 1$ and $d\geq 0$. This approximation $u\in S_{m+d}^{(d)}(Z_N)$ will be determined by collocation methods. The attainable order of global and local convergence of these methods is analyzed in detail.

2. COLLOCATION IN POLYNOMIAL SPLINE SPACES $S_{m+d}^{(d)}(Z_N)$

We shall assume in the following that mesh sequence $(\Pi_N)_{n\geq 1}$ is quasi-uniform, that is, there exists a finite constant γ independent of N such that:

$$\max_{(n)}(h_n)\,/\,\min_n(h_n)\leq \gamma<\infty, \text{ for all } N\in\mathbb{N}.$$

In [7] M. Micula and G. Micula proved that an element $u\in S_{m+d}^{(d)}(Z_N)$ has for all $n=0,\ldots,N-1$ and for all $t\in\sigma_n$ the following form:

$$u(t)=u_n(t)=\sum_{r=0}^{d}\frac{u_{n-1}^{(r)}(t_n)}{r!}(t-t_n)^r+\sum_{r=1}^{m}a_{n,r}(t-t_n)^{d+r}, \tag{2.1}$$

where:

$$u_{-1}^{(r)}(0):=\left[\frac{\mathrm{d}^r}{\mathrm{d}t^r}u(t)\right]_{t=0}=y^{(r)}(0),\ r=0,1,\ldots,d.$$

From (2.1) we have that on element $u\in S_{m+d}^{(d)}(Z_N)$ is well defined when we know the coefficients $\{a_{n,r}\}_{r=\overline{1,m}}$ for all $n=0,\ldots,N-1$. In order to determine these coefficients we consider the set of collocation parameters $\{c_j\}_{j=\overline{1,m}}$, where $0\leq c_1<\ldots<c_m\leq 1$, and we define the set of collocation points by:

$$X(N):=\bigcup_{n=0}^{N-1}X_n,\ \text{with}\ X_n:=\{t_{n,j}:=t_n+c_jh_n,\ j=1,2,\ldots,m\}. \tag{2.2}$$

The approximate solution $u\in S_{m+d}^{(d)}(Z_N)$ will be determined imposing the condition that u satisfy the *VIDE* (1.1) on $X(N)$ and the initial condition, i.e.:

$$u'(t)=f(t,u(t))+\int_0^t K(t,s,u(s))\,\mathrm{d}s,\ \text{for all } t\in X(N),\ \text{with } u(0)=y_0. \tag{2.3}$$

The exact collocation equation (2.3) may be written in the form:

$$u'_n(t_{n,j})=f(t_{n,j},u_n(t_{n,j}))+h_n\int_0^{c_j}K(t_{n,j},t_n+\nu h_n,u_n(t_n+\nu h_n))\mathrm{d}\nu+\overline{F}_n(t_{n,j}), \tag{2.4}$$

$$j=1,\ldots,m(n=0,\ldots,N-1),$$

where:

$$\overline{F}_n(t):=\sum_{i=0}^{n-1}h_i\int_0^1 K(t,t_i+\nu h_i,u_i(t_i+\nu h_i))\,\mathrm{d}\nu$$

denotes the lag term.

For h small enought it is easy to show that system (2.4) has a unique solution $\{a_{n,j}\}_{j=\overline{1,m}}$ for all $n=0,\ldots,N-1$.

For linear the version of (1.1)

$$y'(t)=p(t)y(t)+q(t)+\int_0^t K(t,s)y(s)\mathrm{d}s,\ t\in I,\ y(0)=y_0 \tag{2.5}$$

the collocation equation assumes the form:

$$u'_n(t_{n,j})=p(t_{n,j})u_n(t_{n,j})+q(t_{n,j})+h_n\phi_{n,n}^{(j)}[u_n]+\sum_{i=0}^{n-1}h_i\phi_{n,i}^{(j)}[u_i], \tag{2.6}$$

$$j=1,\ldots,m\ \ (n=0,\ldots,N-1),$$

where:

$$\phi_{n,i}^{(j)}[u_i]=\begin{cases}\int_0^{c_j}K(t_{n,j},t_n+\nu h_n)u_n(t_n+\nu h_n)\mathrm{d}\nu, & \text{if } i=n\\ \int_0^1 K(t_{n,j},t_i+\nu h_i)u_i(t_i+\nu h_i)\mathrm{d}\nu, & \text{if } i=0,\ldots,n-1.\end{cases} \tag{2.7}$$

We phrase our convergence results for the linear equation (2.6); a remark on the extension of these results to the general case (1.1) will follow each of the proofs.

In most applications the integrals (2.7) occurring in the exact collocation (2.6) cannot be evaluated analytically, and one is forced to resort to employing suitable quadrature formulas for their approximation. In the following we suppose that these integrals are approximated by quadrature formulas of the form:

$$(2.8)\quad \hat{\phi}_{n,i}^{(j)}[u_i] := \begin{cases} \sum_{l=1}^{\mu_1} w_l K(t_{n,j}, t_i + d_l h_i) u_i(t_i + d_l h_i) & , \quad \text{if } i = 0,\dots,n-1, \\ \sum_{l=1}^{\mu_0} w_{j,l} K(t_{n,j}, t_n + d_{j,l} h_n) u_n(t_n + d_{j,l} h_n) & , \quad \text{if } i = n; \end{cases}$$

where μ_0 and μ_1 are two given positive integers; $\{d_l\}$, $\{d_{j,l}\}$ are two sets of parameters satisfying, respectively:

$$0 \le d_1 < \dots < d_{\mu_1} \le 1 \text{ and } 0 \le d_{j,1} < \dots < d_{j,\mu_0} \le c_j, \ (j = 1,\dots,m);$$

and w_l, $w_{j,l}$ denote the quadrature weights.

The corresponding error term are defined by:

$$(2.9)\quad E_{n,i}^{(j)}[u_i] = \phi_{n,i}^{(j)}[u_i] - \hat{\phi}_{n,i}^{(j)}[u_i], \quad j = 1,\dots,m, \quad i = 0,\dots,n, \quad (n = 0,\dots,N-1).$$

Hence, the fully discretization version of the collocation equation (2.6) is given by:

$$(2.10)\quad \hat{u}'_n(t_{n,j}) = p(t_{n,j})\hat{u}_n(t_{n,j}) + q(t_{n,j}) + h_n \hat{\phi}_{n,n}^{(j)}[\hat{u}_n] + \sum_{i=0}^{n-1} h_i \hat{\phi}_{n,i}^{(j)}[\hat{u}_i], \quad j = 1,\dots,m, \ (n = 0,\dots,N-1).$$

One can observe that the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$, given by the fully discretized collocation equations (2.10) will, in general, be different from the approximation $u \in S_{m+d}^{(d)}(Z_N)$ given by the exact collocation equations (2.6). For all $n = 0, \dots, N-1$ and for all $t \in \sigma_n$ the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ has the form:

$$(2.11)\quad \hat{u}(t) = \hat{u}_n(t) = \sum_{r=0}^{d} \frac{\hat{u}_{n-1}^{(r)}(t_n)}{r!}(t - t_n)^r + \sum_{r=1}^{m} \hat{a}_{n,r}(t - t_n)^{d+r},$$

with:

$$\hat{u}_{-1}^{(r)}(0) := y^{(r)}(0), \qquad r = 0,1,\dots,d.$$

Equations (2.6) and (2.10) represent, for each $n = 0,1,\dots, N-1$ a recursive system which will give the unknowns $\{a_{n,r}\}_{r=\overline{1,m}}$, respectively $\{\hat{a}_{n,r}\}_{r=\overline{1,m}}$. Since this solutions have been found, the values of u and $\hat{u}$ together with their derivatives on σ_n are determined by the formula (2.1), respectively, by the formula (2.11).

3. GLOBAL CONVERGENCE RESULTS

If the given functions p, q and K are of class $m+d$ on their domain of definition, then the *VIDE* (2.5) has a unique solution y, which is of class $m+d+1$. For a function φ defined on I we shall denote by φ_n the restriction of φ to the subinterval σ_n, for all $n = 0,1, \dots, N-1$, and we shall use the following norm:

$$(3.1)\quad \|\varphi\|_\infty := \sup\{|\varphi_n(t)| : t \in \sigma_n, n = 0,1,\dots, N-1\}.$$

Concerning the convergence of the method described above we give the following theorems:

THEOREM 3.1 *Let p, q and K in (2.5) be $m+d$ times continuously differentiable on their respective domains I and S. Then, for every choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ with $0 < c_1 < c_2 < \dots < c_m \le 1$ and for all quasi-uniform mesh sequences $\{\Pi_N\}$ with sufficiently small $h > 0$, we have:*

(i) the exact collocation equation (2.6) defines a unique approximation $u \in S_{m+d}^{(d)}(Z_N)$, and the resulting error function $e := y - u$ satisfies:

$$(3.2)\quad \left\|e^{(k)}\right\|_\infty \le C_k h^{m+d+1-k}, \quad \textit{for all } k = 0,1,\dots,m+d,$$

where C_k are finite constants independent of h;

(ii) if the quadrature formulas (2.8) satisfy:

$$(3.3)\quad \int_0^1 \phi(t_i + \tau h_i)\,d\tau - \sum_{l=1}^{\mu_1} w_l \phi(t_i + d_l h_i) = O(h_i^{r_1}),$$

and, for j=1, ..., m,

$$(3.4)\quad \int_0^{c_j} \phi(t_n + \tau h_n)\,d\tau - \sum_{l=1}^{\mu_0} w_{j,l} \phi(t_n + d_{j,l} h_n) = O(h_n^{r_0}),$$

whenever the integrand is a sufficiently smooth function, then for the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ defined by the discretized collocation equation (2.10), the following relations hold:

$$(3.5)\qquad \left\|\varepsilon^{(k)}\right\|_{\infty} := \left\|u^{(k)} - \hat{u}^{(k)}\right\|_{\infty} < Q_k h^{s'-k}, \quad \text{for all } k = 0,\ldots,s',$$

and

$$(3.6)\qquad \left\|\hat{e}^{(k)}\right\|_{\infty} := \left\|y^{(k)} - \hat{u}^{(k)}\right\|_{\infty} < \hat{C}_k h^{s-k}, \quad \text{for all } k = 0,\ldots,s,$$

where $s' = \min\{r_0 + 1, r_1\} + 1$, $s = \min\{s', m + d + 1\}$ *and* $Q_k, \hat{C}_k$ *are finite constants independent of* h.

Proof. We shall prove it by induction using the same technique as in [4] or in [5].

(i) For $n=0,1,\ldots,N-1$ and for all $t = t_n + \tau h_n \in \sigma_n$ $(\tau \in (0,1])$ the exact solution y can be developed in Taylor series:

$$(3.7)\qquad y(t_n + \tau h_n) = \sum_{r=1}^{m+d} \frac{y^{(r)}(t_n)}{r!}\tau^r h_n^r + h_n^{m+d+1} R_n(\tau),$$

where:

$$R_n(\tau) = \frac{1}{(m+d)!}\int_0^{\tau} y^{(m+d+1)}(t_n + \eta h_n)(\tau - \eta)^{m+d}\,\mathrm{d}\eta.$$

So, by (2.1) and (3.7) we have:

$$(3.8)\qquad e_n(t_n + \tau h_n) = \sum_{r=0}^{d} \frac{e_{n-1}^{(r)}}{r!} h_n^r \tau^r + h_n^p \sum_{r=1}^{m} \beta_{n,r}\tau^{d+r} + h_n^{m+d+1} R_n(\tau),$$

where:

$$h_n^p \beta_{n,r} = \left(\frac{y^{(d+r)}(t_n) - a_{n,r}}{(d+r)!}\right) h_n^{d+r}.$$

Taking into account that y is a solution of *VIDE* (2.5) and $u \in S_{m+d}^{(d)}(Z_N)$ satisfies the exact collocation equation (2.6) and employing the expression (3.8) for e_n, we are led to:

$$h_n^{p-1}\sum_{r=1}^{m}\beta_{n,r}\left\{(d+r)c_j^{d+r-1} - h_n c_j^{d+r} p_{n,j} - h_n^2\int_0^{c_j} k_{n,j}(t_n + \tau h_n)\tau^{d+r}\mathrm{d}\tau\right\} =$$

$$(3.9)\qquad = \sum_{r=0}^{d}\frac{e_{n-1}^{(r)}(t_n)}{r!} h_n^{r-1}\left(-rc_j^{r-1} + h_n p_{n,j} c_j^r + h_n^2\int_0^{c_j} k_{n,j}(t_n + \tau h_n)\tau^r\mathrm{d}\tau\right) +$$

$$+h_n^{m+d}\left\{-R'_n(c_j) + p_{n,j} h_n R_n(c_j) + h_n^2\int_0^{c_j} k_{n,j}(t_n + \tau h_n) R_n(\tau)\mathrm{d}\tau\right\} +$$

$$+\sum_{i=0}^{n-1} h_i \int_0^1 k_{n,j}(t_i + \tau h_i) e_i(t_i + \tau h_i)\mathrm{d}\tau, \quad j = 1,\ldots,m,$$

where, we have introduced the abbreviations $p_{n,j} := p(t_n + c_j h_n)$ and $k_{n,j}(\cdot) = K(t_{n,j}, \cdot)$.

Relation (3.9) can be written:

$$(3.10)\qquad h_n^{p-1} D_n \beta_n = F_n \cdot E_n + h_n^{m+d} r_n + \sum_{i=0}^{n-1} h_i q_{n,i},$$

where $D_n \in \mathcal{M}_{m\times m}$, $F_n \in \mathcal{M}_{m\times(d+1)}$, $E_n \in \mathcal{M}_{(d+1)\times(d+1)}$ ($\mathcal{M}_{\alpha\times\gamma}$ denote the set of matrices with α lines and γ columns) and β_n, r_n, $q_{n,i}$ are the column vectors. The explicit form of the matrices and the vectors results from (3.9).

For $n=0$, by (2.1) and (3.10) we obtain:

$$(3.11)\qquad h_0^{p-1} D_0 \beta_0 = h_0^{m+d} r_0.$$

From the assumptions of the theorem it results that the vector r_0 is bounded and for sufficiently small $h_0 > 0$ the matrix D_0 possesses a uniformly bounded inverse. Hence, for $p = m + d + 1$ we have:

$$(3.12)\qquad \|\beta_0\|_1 := \sum_{l=1}^{m}|\beta_{0,l}| \le \left\|D_0^{-1}\right\|_1 \|r_0\|_1 =: M_0,$$

and from (3.8) it results:

$$(3.13)\qquad |e_0(t_0 + \tau h_0)| \le h^{m+d+1}\left(M_0 + |R_0(\tau)|\right) \le C_0 \cdot h^{m+d+1}, \quad \text{for all } \tau \in [0,1].$$

Deriving relation (3.8) k times ($k = 1, 2, \ldots, m+d$) and using (3.12) we obtain:

$$(3.14)\qquad \left|e_0^{(k)}(t_0 + \tau h_0)\right| \le C_0^{(k)} \cdot h^{m+d+1-k}, \quad \text{for all } \tau \in [0,1].$$

Suppose now that, for all $j = 0, 1, \ldots, n-1$

$$(3.15)\qquad \left|e_j^{(k)}(t_j + \tau h_j)\right| \le C_j^{(k)} \cdot h^{m+d+1-k}, \quad \tau \in (0,1], \quad k = 0,\ldots,m+d$$

hold and prove that (3.15) holds for $j = n$.

By (3.9), (3.10), (3.15) and the assumptions of the theorem it follows that for sufficiently small $h_n > 0$ the matrix D_n possess a uniformly bounded inverse, $\|E_n\|_1 = O\left(h^{m+d+1}\right)$, $\|q_{n,i}\|_1 = O\left(h^{m+d+1}\right)$ $(i = 0,1,\ldots,n-1)$ and $\|r_n\|_1$ is bound.

Thus, from (3.10) for $p = m + d + 1$, we obtain:

$$(3.16)\qquad \|\beta_n\|_1 := \sum_{l=1}^{m}|\beta_{n,l}| \le M_n + M'_n h,$$

and from (3.8) it results:

(3.17) $$\left|e_n^{(k)}(t_n + \tau h_n)\right| \le C_n^{(k)} h_n^{m+d+1-k},$$

for all $\tau \in (0,1]$ and $k = 0,1,\dots,m+d$.

Evaluations (3.14), (3.15) and (3.17) end the proof of the first assertion by the theorem.

(ii) By (2.1) and (2.11) it follows that the function $\varepsilon := u - \hat{u}$ can be written for every $n = 0,1,\dots,N-1$, thus:

(3.18) $$\varepsilon_n(t_n + \tau h_n) = \sum_{r=0}^{d} \frac{\varepsilon_{n-1}^{(r)}(t_n)}{r!} \tau^r h_n^r + h_n^{s'} \sum_{r=1}^{m} \eta_{n,r} \tau^{d+r},$$

where:
$$h_n^{s'} \eta_{n,r} := \left(a_{n,r} - \hat{a}_{n,r}\right) h_n^{d+r}.$$

If we now substract the discretized collocation equation (2.10) from the exact collocation equation (2.6) and we use relations (2.9) and (3.18), we are led to:

(3.19) $$h_n^{s'-1} \hat{D}_n \eta_n = \hat{F}_n \mathscr{E}_n + h_n r_{n,n} + \sum_{i=0}^{n-1} h_i \left(\hat{q}_{n,i} + r_{n,i}\right),$$

where $r_{n,i} := \left(E_{n,i}^{(1)}[u_i], \dots, E_{n,i}^{(m)}[u_i]\right)$, and the matrices $\hat{D}_n$, $\hat{F}_n$, $\mathscr{E}_n$ and the vectors $\hat{q}_{n,i}$ have the same sizes as D_n, F_n, E_n and $q_{n,i}$ from (3.10), the differences between them consisting in the fact that the integrals from (3.9) are replaced with quadrature formulas of the form (2.8).

The above expression has the same structure as (3.10). From the smoothness hypothesis and from the assumptions on the order of the quadrature formulas (3.3) and (3.4) we have: $\|r_{n,n}\|_1 = O(h_n^{r_0})$ and $\|r_{n,i}\|_1 = O(h_i^{r_i})$. Thus, repeating the reasoning from the proof of the assertion (i), it easily results that relation (3.5) is true.

Now by (3.2) and (3.5) it results:

$$\left\|\hat{e}^{(k)}\right\|_\infty := \left\|y^{(k)} - \hat{u}^{(k)}\right\|_\infty \le \left\|e^{(k)}\right\|_\infty + \left\|\varepsilon^{(k)}\right\|_\infty \le \hat{C}_k h^{s-k}$$

for all $k = 0,1,\dots,s$, with $s = \min\{s', m+d+1\}$.

COROLLARY 3.2 *Let the assumptions of Theorem 3.1 hold. If the quadrature formulas (2.8) are of interpolatory type, with $\mu_0 = \mu_1 = m+d$, then the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ defined by the discretized collocation equation (2.10) leads to an error $\hat{e}(t)$ satisfying:*

(3.20) $$\left\|\hat{e}^{(k)}\right\|_\infty = O\left(h^{m+d+1-k}\right), \quad (as\ h\searrow 0\ \ and\ \ Nh \le \gamma T),$$

for $k = 0, \dots, m+d$, and for every choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ with $0 < c_1 < \dots < c_m \le 1$.

In many papers (see [1], [2], [3]) the quadrature formulas used have $\mu_0 = \mu_1 = m$, $d_j = c_j$ and $d_{j,l} = c_j c_l$ ($j,l = 1,\dots,m$). The possibility of employing some quadrature formulas of the this type in our method would lead to some simplifications. These simplifications are useful when they do not spoil the convergence order given by Theorem 3.1 (i), namely $s = m+d+1$. An answer to this problem is given in the following corollary.

COROLLARY 3.3. *If in VIDE (2.5), $p \in C^{m+d}(I)$, $q \in C^{m+d}(I)$ and $K \in C^{m+d}(S)$ and if $m \ge d$, then there exists the set of collocation parameters $\{c_j\}_{j=\overline{1,m}}$ such that for the approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ given by the discrete collocation equations (2.10) in which $\mu_0 = \mu_1 = m$, $d_j = c_j$ and $d_{j,l} = c_j c_l$ we have:*

(3.21) $$\left\|\hat{e}^{(k)}\right\|_\infty := \left\|y^{(k)} - \hat{u}^{(k)}\right\|_\infty = O\left(h^{m+d+1-k}\right),$$

for $k = 0,\dots,m+d$ (as $h \searrow 0$ with $Nh \le \gamma T$).

Proof. If $\mu_0 = \mu_1 = m$ and $m \ge d$ then we choose the set of collocation parameters $\{c_j\}_{j=\overline{1,m}}$ to be formed by the m Gauss points for $(0,1)$.

Remark 3.4. (i) The results of the above theorems for $d = 0$ and $m \ge 1$ are similar to the results given in [3], while for $d = n-1$ and $m = 1$ ($n \in N$) they are similar to the result from [6].

(ii) The extension of the above arguments to nonlinear *VIDE* (1.1) is straightforward: in the error equations (3.10) and (3.19), the roles of $p_{n,j}$ and of $k_{n,j}(t_i + \tau h_i)$ are taken, respectively, by $\partial f(t_n + c_j h_n, z_{n,j})/\partial y$ and $\partial K(t_n + c_j h_n, t_i + \tau h_i, z_i(\tau))/\partial y$, with $z_{n,j}$ and $z_i(\tau)$ denoting suitable intermediate values arising in the application of the Mean-Value Theorem (see [3], [5]).

4. LOCAL SUPERCONVERGENCE ON Z_N

The notion of local superconvergence is used when on a set of interior points Z_N (or $\overline{Z}_N$), the approximate solution has a convergence order greater than the global convergence order. From Theorem 3.1 we notice that the only conditions imposed on the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ are that they must be distinct and they must belong to (0,1]. The local superconvergence on $\overline{Z}_N$ is closely connected with the choice of the collocation parameters (see [3], [4], [5]) and with the relation between their number and the number of the coefficients of the approximate solution determined from the smooth conditions.

We will give the following theorem concerning the aspects presented above:

THEOREM 4.1. *Suppose that:*

(I) *the given functions p, g and K from VIDE (2.5) are m+p times continuously differentiable on their respective domains I and S (where $d+1<p\le m$);*

(II) $m \ge d+2$;

(III) *the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$, *with* $0<c_1<\ldots<c_m\le 1$ *are chosen such that:*

$$J_k := \int_0^1 s^k \prod_{j=1}^{m}(s-c_j)\mathrm{d}s = 0, \quad for \quad k=0,1,\ldots,p-1;$$

(4.1)

$$J_p \neq 0, \qquad where \quad d+1<p\le m.$$

Then, for all quasi-uniform mesh sequences $\{\Pi_N\}$ *with sufficiently small* $h>0$, *we have:*

(i) *if* $u\in S_{m+d}^{(d)}(Z_N)$ *is the approximate solution defined by the exact collocation equation (2.6) and y is the exact solution of VIDE (2.5) then:*

(4.2) $$\max_{t_n\in\overline{Z}_N}|y(t_n)-u(t_n)| = O(h^{m+p}), \quad (as\ h\searrow 0 \ \ and \ \ Nh\le\gamma T);$$

(ii) *if the quadrature formulas (2.8) satisfy (3.3) and* $\hat{u}\in S_{m+d}^{(d)}(Z_N)$ *is the approximate solution defined by the discretized collocation equation (2.10) then:*

(4.3) $$\max_{t_n\in\overline{Z}_N}|y(t_n)-\hat{u}(t_n)| = O(h^{\alpha}), \quad (as\ h\searrow 0 \ \ and \ \ Nh\le\gamma T),$$

where $\alpha=\min\{m+p, s'\}$;

(iii) if $m\ge d+2$ *and the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$, *are chosen such that relation (4.1) holds and* $c_m=1$, *then:*

(4.4) $$\max_{t_n\in\overline{Z}_N}|y'(t_n)-u'(t_n)| = O(h^{m+p}), \quad (as\ h\searrow 0 \ \ and \ \ Nh\le\gamma T),$$

and

(4.5) $$\max_{t_n\in\overline{Z}_N}|y'(t_n)-\hat{u}'(t_n)| = O(h^{\alpha}), \ (as\ h\searrow 0 \ \ and \ \ Nh\le\gamma T),$$

where $\alpha=\min\{m+p, s'\}$.

Proof. (i) The exact collocation equation (2.6) can be written in the form:

(4.6) $$u'(t)=q(t)+p(t)u(t)+\int_0^t K(t,s)u(s)\mathrm{d}s-\delta(t), t\in I,$$

where $\delta(t)$ denotes a suitable function, subsequently called the defect function, vanishing on $X(N)$.

By (4.6) and (2.5) we obtain for the error function $e:=y-u$ the following *VIDE*:

(4.7) $$e'(t) = \delta(t) + e(t)\delta(t) + \int_0^t K(t,s)e(s)\mathrm{d}s, \ \ t\in I \ \ \text{with} \ \ e(0)=0.$$

The solution of (4.7) can be expressed in the form (see Theorem 1.3.4. from [3]):

(4.8) $$e(t) = R(t,0)e(0) + \int_0^t R(t,s)\delta(s)\mathrm{d}s = \int_0^t R(t,s)\delta(s)\mathrm{d}s,$$

where $R(t,s)$ represents the resolvent kernel associated with the *VIDE* (2.5), and hence with *VIDE* (4.7).

If in (4.8), for $t=t_n\in\overline{Z}_N$, we replace each integral by the sum of the interpolatory quadrature formula with abscissas $\{t_i+c_lh_i : l=1,\ldots,m\}$ and the corresponding remainder term $E_{n,i}$; since $\delta(t_i+c_lh_i)=0$, we obtain:

(4.9) $$e(t_n)=\sum_{i=0}^{n-1}h_iE_{n,i}:=\sum_{i=0}^{n-1}h_i\int_0^1 R(t_n,s)\delta(s)\mathrm{d}s.$$

From (4.1) we have that for $|E_{n,i}|=O(h^{m+p})$ for $h\to 0$ and hence from (4.9) it results $|e(t_n)|=O(h^{m+p})$, evaluation which proves the first assertion of the theorem.

(ii) The assertion of Theorem 4.1 (ii) now follows from (3.5) and (4.2). We mention that relation (4.3) can be straightly proved using the same technique as in Theorem 3.1 from [5].

(iii)By (4.8) we obtain:

(4.10) $$e'(t) = \delta(t)R(t,t) + \int_0^t \frac{\partial R(t,s)}{\partial t}\delta(s)\mathrm{d}s = \delta(t) + \int_0^t \frac{\partial R(t,s)}{\partial t}\delta(s)\mathrm{d}s,$$

since $R(t,t)=1$ (see T 1.3.4 from [3]).

For $0<c_1<\ldots<c_m=1$ we have $\delta(t_n)=0$ and by (4.10) for $t=t_n\in Z_n$ it results, in complete analogy to (4.9):

(4.11) $$e'(t_n)=\delta(t_n)+\sum_{i=0}^{n-1}h_i\overline{E}_{n,i}=\sum_{i=0}^{n-1}h_i\overline{E}_{n,i},$$

where $\overline{E}_{n,i}$ denote the quadrature errors associated with the m-point interpolatory quadrature formulas, based on the abscissas $\{t_i + c_j h_i\}$, for the integrals from (4.10). The assertions of Theorem (3.1) (iii) now follows by the arguments employed at the end of the proof of (i) and (ii).

COROLLARY 4.2. *Let the assumptions of Theorem 3.1 hold. Then:*

(i) *for the approximation* $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ *given by the discrete collocation equation* (2.10) *in which* $\mu_0 = \mu_1 = m$, $d_j = c_j$ *and* $d_{j,l} = c_j c_l$ *we have:*

$$\max_{t_n \in \overline{Z}_N} |y(t_n) - \hat{u}(t_n)| = O(h^{m+p}) \quad (as\ h \searrow 0\ \ and\ \ Nh \le \gamma T),$$

and for $c_m = 1$ *we have:*

$$\max_{t_n \in \overline{Z}_N} |y'(t_n) - \hat{u}'(t_n)| = O(h^{m+p}) \quad (as\ h \searrow 0\ \ and\ \ Nh \le \gamma T);$$

(ii) *if the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the zeros of* $P_m(2s-1)$ *(Gauss points for* $(0,1)$*), then* $p = m$ *and*

$$\max_{t_n \in \overline{Z}_N} |e(t_n)| = O(h^{2m}) \quad (as\ h \searrow 0\ \ and\ \ Nh \le \gamma T);$$

(iii) *if the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the zeros of* $P_{m-1}(2s-1) - P_m(2s-1)$ *(Radau II points for* $(0,1]$*), then* $p = m-1$ *and*

$$\max_{t_n \in \overline{Z}_N} |e^{(i)}(t_n)| = O(h^{2m-1}), \quad for \quad i = 0,1 \ (as\ h \searrow 0\ \ and\ \ Nh \le \gamma T);$$

(iv) *if the discretized collocation equation* (2.10) *is characterized by interpolatory m-point qaadrature approximations with* $\mu_0 = \mu_1 = m$, *then the resulting approximation* $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ *has the property that:*

$$\max_{t_n \in \overline{Z}_N} |\hat{e}(t_n)| = O(h^{2m}) \quad (as\ h \searrow 0\ \ and\ \ Nh \le \gamma T),$$

if and only if, (a), (b) *and one of* (c), (c'), (c'') *holds:*

(a) *the collocation parameters* $\{c_j\}_{j=\overline{1,m}}$ *are the Gauss points for* $(0,1)$;

(b) $d_l = c_l$ $(l = 1, \ldots, m)$;

(c) $d_{j,l} = c_j c_l$ $(j,l = 1, \ldots, m)$;

(c') $d_{j,l} = c_j c'_l$ $(j, l = 1, \ldots, m)$, *where the* $\{c'_j\}_{j=\overline{1,m}}$ *are the Radau I points for* $[0,1)$;

(c'') $d_{j,l} = c_j c''_l$ $(j,l = 1, \ldots, m)$, *where the* $\{c''_j\}_{j=\overline{1,m}}$ *are the Radau II points for* $(0,1]$.

Proof. The above results are proved by H. Brunner and P. J. van der Houwen in the case $d=0$ (i..e $S_m^{(0)}(Z_N)$ (see [3], pp.279–299)). Also they hold in our case ($d \ge 0$), the proofs can be identically transposed.

5. NUMERICAL EXAMPLES

The convergence results derived in the preceding sections will be illustrated by the collocation methods to the following test problem:

$$(5.1) \qquad y'(t) = \lambda \int_0^t y(s)ds, \quad y(0) = 1, \quad t \in [0,1], \quad \lambda > 0,$$

whose exact solution is $y(t) = \frac{1}{2}\left(\exp\left(\sqrt{\lambda t}\right) + \exp\left(-\sqrt{\lambda t}\right)\right)$, and two linear problems:

$$(5.2) \qquad \begin{aligned} & y'(t) = y(t) + 2t\exp(t^2) + \int_0^t 2t\exp(t^2 - s^2)y(s)ds, \\ & y(0) = 1, \quad t \in [0,1], \end{aligned}$$

whose exact solution is $y(t) = \exp(t + t^2)$, and

$$(5.3) \qquad \begin{aligned} & y'(t) = -y(t) + \exp(t) - \int_0^t \exp(t-s)y(s)ds, \\ & y(0) = 1, \quad t \in [0,1], \end{aligned}$$

whose exact solution is $y(t) = 1$.

For above problems we have tested the collocation methods based on:

A. set of collocation points $\left\{c_1 = \frac{1}{2}, c_2 = 1\right\}$ if $m = 2$, and the set $\left\{c_1 = \frac{1}{3}, c_2 = \frac{1}{2}, c_3 = 1\right\}$ if $m = 3$;

B. Radau II points $\left\{c_1 = \frac{1}{3}, c_2 = 1\right\}$ if $m = 2$, and the points $\left\{c_1 = \frac{4-\sqrt{6}}{10}, c_2 = \frac{4+\sqrt{6}}{10}, c_3 = 1\right\}$ if $m = 3$;

C. Gauss points $\left\{c_1=\frac{3-\sqrt{3}}{6}, c_2=\frac{3+\sqrt{3}}{6}\right\}$ if $m = 2$, and the points $\left\{c_1=\frac{5-\sqrt{15}}{10}, c_2=\frac{1}{2}, c_3=\frac{5+\sqrt{15}}{10}\right\}$ if $m = 3$.

The tables contain the values of approximated error in the end point, i.e. the value $e_N = |y(T)-u(T)|$, the number of correct digits obtained at the end point, i.e. the value of:

$$sd := -\log_{10}\left(\frac{|y(T)-u(T)|}{|y(T)|}\right), (T = t_N)$$

and the effective order of numerical method, i.e. the value of

$$p_{eff} := \frac{sd(h) - sd(2h)}{\log_{10}(2)}, \qquad (\log_{10}(2) \approx 0.3)$$

for various values of the *h, m, d.*

Table 5.1.a

Approximated error, number of correct significant digits and effective orders for problem (5.1), with $\lambda = 1$, for $m = 2$ and $d = 0$

h	*A* e_N / sd	*A* p_{eef}	*B* e_N / sd	*B* p_{eef}	*C* e_N / sd	*C* p_{eef}
1/2	$.68\times10^{-2}$ / 2.35		$.37\times10^{-3}$ / 3.61		$.17\times10^{-4}$ / 4.94	
		2.1		2.8		4.06
1/4	$.16\times10^{-2}$ / 2.98		$.54\times10^{-4}$ / 4.45		$.10\times10^{-5}$ / 6.16	
		2.03		2.9		4
1/8	$.39\times10^{-3}$ / 3.59		$.70\times10^{-5}$ / 5.33		$.67\times10^{-7}$ / 7.36	
		2.03		3		4.06
1/16	$.96\times10^{-4}$ / 4.20		$.89\times10^{-6}$ / 6.23		$.40\times10^{-8}$ / 8.58	

Table 5.1.b

Approximated error, number of correct significant digits and effective orders for problem (5.1), with $\lambda = 1$, for $m = 3$ and $d = 0$

h	*A* e_N / sd	*A* p_{eef}	*B* e_N / sd	*B* p_{eef}	*C* e_N / sd	*C* p_{eef}
1/2	$.83\times10^{-3}$ / 3.26		$.54\times10^{-5}$ / 5.45		$.23\times10^{-7}$ / 7.82	
		3.06		5		6.03
1/4	$.10\times10^{-3}$ / 4.18		$.17\times10^{-6}$ / 6.95		$.35\times10^{-9}$ / 9.63	
		3		5.5		6.03
1/8	$.12\times10^{-4}$ / 5.08		$.40\times10^{-8}$ / 8.58		$.56\times10^{-11}$ / 11.44	

Table 5.1.c

Approximated error, number of correct significant digits and effective orders for problem (5.1), with $\lambda = 1$, for $m = 3$ and $d = 1$

h	*A* e_N / sd	*A* p_{eef}	*B* e_N / sd	*B* p_{eef}	*C* e_N / sd	*C* p_{eef}
1/2	$.12\times10^{-4}$ / 5.08		$.20\times10^{-6}$ / 6.87		$.41\times10^{-7}$ / 7.57	
		4.03		5		4.4
1/4	$.78\times10^{-6}$ / 6.29		$.66\times10^{-8}$ / 8.37		$.20\times10^{-8}$ / 8.88	
		4		5.03		4.03
1/8	$.50\times10^{-7}$ / 7.49		$.20\times10^{-9}$ / 9.88		$.12\times10^{-9}$ / 10.09	

Table 5.2

Approximated error, number of correct significant digits and effective orders for problem (5.2), for $m = 2$ and $d = 1$

h	*A* e_N / sd	*A* p_{eef}	*B* e_N / sd	*B* p_{eef}	*C* e_N / sd	*C* p_{eef}
1/2	$.27\times10^{-1}$ / 2.43		2.1×10^{-1} / 1.52		$.48\times10^{-2}$ / 3.18	
		4.3		2.86		4.01
1/4	$.15\times10^{-2}$ / 3.69		$.30\times10^{-1}$ / 2.38		$.28\times10^{-3}$ / 4.41	
		4.06		2.93		4
1/8	$.90\times10^{-4}$ / 4.91		$.39\times10^{-2}$ / 3.26		$.17\times10^{-7}$ / 5.61	
		4.03		3		4.03
1/16	$.55\times10^{-5}$ / 6.12		$.51\times10^{-3}$ / 4.16		$.10\times10^{-5}$ / 6.83	

Using the Maple Programing Language, the collocation method apply at the problem (5.3) in all cases from above, we yield the exact solution, i.e. $u(t)=y(t)$ for all $t\in[0,1]$.

Finally, from numerical examples printed in Tables 5.1 and 5.2, we can observe a good concordance between theoretical results presented in the preceding sections and corresponding results given in this section.

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