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A CHARACTERIZATION OF A GENERALIZED INTEGRAL MEAN PRESERVING CONVEXITY

ELEONORA DAPPA
(Częstochowa, Poland)

An operator

$$f \mapsto F(f),$$

given by the formula:

$$F(f)(x) := \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{n}{x^n} \displaystyle\int_0^x t^{n-1} f(t)\,\mathrm{d}t & \text{for } x \neq 0 \end{cases}$$

transforms the class $K(b)$, $b > 0$, all real functions, convex on $[0, b]$ and vanishing at zero, into itself. In particular, for $n := 1$ it reduces itself to the usual integral mean. More generally, given a suitable function φ (instead of "$x \mapsto x^n$") one may consider an integral operator:

(1) $$F_\varphi(f)(x) := \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{n}{\varphi(x)} \displaystyle\int_0^x \varphi'(t) f(t)\,\mathrm{d}t & \text{for } x \neq 0. \end{cases}$$

This operator was considered by Gh. Toader in [1]. He has proved that the inclusion $F_\varphi(K(b)) \subset K(b)$ forces φ to be proportional to a power function. More exactly:

THEOREM (Gh. Toader [1]). *If the operator F_φ given by (1) on the class $K(b)$ preserves the convexity, then there exist a real constant $k > 0$ and an $a > 0$ such that $\varphi(t) = kt^a$, $t \in [0, b]$. Conversely, if $\varphi(t) = kt^a$, $t \in [0, b]$, $a > 0$, $k \neq 0$, then the operator F_φ given by (1) transforms the class $K(b)$ into itself.*

The chief concern of the present paper is to replace the derivative φ' of the function φ in Toader's operator by another given function. More precisely, we shall replace φ' by an arbitrary positive continuous function. So, we shall consider a pair (φ, ψ) of continuous functions enjoying the following properties: $\varphi, \psi : [0, b] \to \mathbb{R}$, $\varphi(0) = 0$, $\varphi(x) \neq 0$ for $x \in (0, b]$. In this way we define an integral mean $F_{\varphi,\psi}$ on the space $C(b)$ of all continuous real functions on $[0, b]$, vanishing at zero with the aid of the formula

$$(2) \qquad F_{\varphi,\psi}(f)(x) := \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{1}{\varphi(x)} \displaystyle\int_0^x \psi(t) f(t) \mathrm{d}t & \text{for } x \in (0, b] \end{cases}$$

for $f \in C(b)$.

We will prove a result similar to that due to Gh. Toader in [1] for the integral mean $F_{\varphi,\psi}$ just introduced. We shall first show that the operator $F_{\varphi,\psi}$ given by (2) can be represented as the sum of two new operators depending upon one given function only and that one of them is just the Toader operator.

THEOREM 1. *Let* $\varphi, \psi : [0, b] \to \mathbb{R}$ *be continuous functions enjoying the following properties*: $\varphi(0) = 0$, $\varphi(x) > 0$ *and* $\psi(x) > 0$ *for* $x \in (0, b]$ *and let* φ *be (right-hand side) differentiable at zero. If the operator* $F_{\varphi,\psi} : C(b) \to \mathbb{R}^{[0,b]}$, *given by (2) transforms the class* $K(b)$ *into itself, then* φ *is a* C^1*–function on the interval* $(0, b]$ *and there exist a constant* $\gamma > 0$ *such that*

$$F_{\varphi,\psi} = \gamma F_\varphi^1 + \gamma F_\varphi,$$

where

$$(3) \qquad F_\varphi^1(f)(x) := \begin{cases} 0 & \text{for } x = 0 \\ \dfrac{1}{\varphi(x)} \displaystyle\int_0^x \frac{\varphi(t)}{t} f(t) \mathrm{d}t & \text{for } x \in (0, b], \end{cases}$$

and the operator F_φ *is given by* (1).

Proof. The identity function $\mathrm{id}(x) = x$, $x \in [0, b]$, belongs to $K(b)$ and, therefore, the function $F_{\varphi,\psi}(\mathrm{id})$ is convex. In particular, for an arbitrary $\lambda \in (0,1)$ and for every $x \in [0, b]$ we have:

$$F_{\varphi,\psi}(\mathrm{id})(\lambda x) = F_{\varphi,\psi}(\mathrm{id})(\lambda x + (1-\lambda)0) \leq \lambda F_{\varphi,\psi}(\mathrm{id})(x) + \\ +(1-\lambda)F_{\varphi,\psi}(\mathrm{id})(0) = \lambda F_{\varphi,\psi}(\mathrm{id})(x),$$

whence, by definition (2), we get the inequality

$$\frac{1}{\varphi(\lambda x)} \int_0^{\lambda x} t\psi(t) \mathrm{d}t \leq \lambda \frac{1}{\varphi(x)} \int_0^x t\psi(t) \mathrm{d}t$$

valid for all $\lambda \in (0,1)$ and $x \in (0, b]$. Similarly, since $-\mathrm{id} \in K(b)$ as well, we obtain the reverse inequality. Consequently,

$$(4) \qquad \frac{1}{\varphi(\lambda x)} \int_0^{\lambda x} t\psi(t) \mathrm{d}t = \lambda \frac{1}{\varphi(x)} \int_0^x t\psi(t) \mathrm{d}t$$

for all $\lambda \in (0,1)$ and $x \in (0, b]$. In particular, for $x = b$ and $\beta := \int_0^b t\psi(t) \mathrm{d}t > 0$ equality (4) assumes the form

$$\frac{1}{\varphi(\lambda b)} \int_0^{\lambda b} t\psi(t) \mathrm{d}t = \frac{\lambda}{\varphi(b)} \beta \qquad \text{for} \quad \lambda \in (0,1).$$

Setting here $s := \lambda b$ we get

$$\frac{1}{\varphi(s)} \int_0^s t\psi(t) \mathrm{d}t = \frac{s}{b\varphi(b)} \beta \qquad \text{for} \quad s \in (0, b],$$

or, equivalently,

$$\varphi(s) = \frac{b\varphi(b)}{\beta} \frac{1}{s} \int_0^s t\psi(t) \mathrm{d}t \qquad \text{for} \quad s \in (0, b].$$

Putting here $\gamma := \dfrac{\beta}{b\varphi(b)}$ we have $\gamma > 0$ and

$$\gamma s \varphi(s) = \int_0^s t\psi(t) \mathrm{d}t \qquad \text{for} \quad s \in (0,1].$$

Moreover, φ is a C^1 –function on $(0, b]$ because ψ is continuous. From the last equality we infer that

$$\psi(s) = \gamma \frac{\varphi(s)}{s} + \gamma\varphi'(s) \qquad \text{for} \quad s \in (0, b].$$

Substituting this equality into (2) gives

$$F_{\varphi,\psi}(f)(s) = \frac{1}{\varphi(x)} \int_0^x f(t) \left[\gamma \frac{\varphi(t)}{t} + \gamma\varphi'(t) \right] \mathrm{d}t = \gamma \frac{1}{\varphi(x)} \int_0^x f(t) \frac{\varphi(t)}{t} \mathrm{d}t + \\ +\gamma \frac{1}{\varphi(x)} \int_0^x f(t)\varphi'(t) \mathrm{d}t = \gamma F_\varphi^1(f)(x) + \gamma F_\varphi(f)(x), \quad x \in (0, b],$$

which completes the proof.

In what follows, we shall show that under some additional assumptions upon the given functions the demand that the corresponding operator $F_{\varphi,\psi}$ transforms the class $K(b)$ into itself, determines the analytic form of φ and ψ. Namely, we have the following

THEOREM 2. *Let* $\varphi, \psi : [0, b] \to \mathbb{R}$, *be continuous functions such that* $\varphi(0) = 0$, $\varphi(x) > 0$ *and* $\psi(x) > 0$ *for* $x \in (0, b]$ *and let* φ *be (right-hand side) differentiable at zero. Assume that there exists a* $c \geq 2$ *such that the function*

(S) $$(0,b] \ni x \mapsto \frac{\psi(x)}{x^{c-2}} \in \mathbb{R} \quad \text{is starshaped}$$

and the function

(M) $$(0,b] \ni x \mapsto \frac{\psi(x)}{x^{c}} \in \mathbb{R} \quad \text{attains the absolute minimum at } b.$$

If the operator $F_{\varphi,\psi} : C(b) \to \mathbb{R}^{[0,b]}$, *given by* (2) *transforms the class* $K(b)$ *into itself, then there exist a real constant* $k > 0$ *and an* $a \in (0, c)$ *such that*

$$\varphi(x) = \varphi_{a,k}(x) := kx^a, \ \psi(x) = \psi_{a,k}(x) = \varphi'_{a,k}(x) = kax^{a-1}, \quad x \in [0, b].$$

Proof. Observe that

$$\frac{1}{\varphi(\lambda x)}\int_0^{\lambda x} t\psi(t)\,dt = \lambda\frac{1}{\varphi(x)}\int_0^x t\psi(t)\,dt \quad \text{for all} \quad \lambda \in (0,1) \quad \text{and} \quad x \in (0,b]$$

(see the proof of Theorem 1). With the aid of the substitution: $t = \lambda u$ (for fixed $\lambda \in (0,1)$ and $x \in (0, b]$) we obtain the equality

$$\frac{\lambda}{\varphi(\lambda x)}\int_0^x t\psi(\lambda t)\,dt = \frac{1}{\varphi(x)}\int_0^x t\psi(t)\,dt$$

valid for all $\lambda \in (0,1)$ and $x \in (0, b]$. Consequently, one has

(5) $$\int_0^x \left[\frac{t\lambda}{\varphi(\lambda x)}\psi(\lambda t) - \frac{t}{\varphi(x)}\psi(t)\right]dt = 0 \quad \text{for all} \quad \lambda \in (0,1) \quad \text{and} \quad x \in (0,b].$$

In particular, for $x := b$ we have

$$\int_0^b \left[\frac{t\lambda}{\varphi(\lambda b)}\psi(\lambda t) - \frac{t}{\varphi(b)}\psi(t)\right]dt = 0\,.$$

We shall show that the expression occurring here under the integral sign is nonpositive. In fact, by assumption we have $\varphi(\lambda b) \geq \lambda^c \varphi(b)$ for all $\lambda \in [0,1]$ whence, in view of (S) and (M) applied consecutively,

$$\frac{t\lambda}{\varphi(\lambda b)}\psi(\lambda t) - \frac{t}{\varphi(b)}\psi(t) \leq t\left[\frac{1}{\varphi(\lambda b)}\lambda^c - \frac{1}{\varphi(b)}\right]\psi(t) \leq 0,$$

as claimed.

Consequently, on account of (5), the difference considered has to vanish identically. In particular, setting $x = b$ we obtain

$$\frac{\lambda b}{\varphi(\lambda b)}\psi(\lambda b) = \frac{b}{\varphi(b)}\psi(b) =: \eta > 0 \quad \text{for} \quad \lambda \in (0,1].$$

Therefore

$$\psi(t) = \eta\frac{\varphi(t)}{t} \qquad \text{for all } t \in (0,b].$$

On the other hand

$$\psi(x) = \gamma\frac{\varphi(x)}{x} + \gamma\varphi'(x), \qquad x \in (0,b],$$

(cf. the proof of Theorem 1) whence

$$\varphi'(x) = \frac{\eta-\gamma}{\gamma}\frac{\varphi(x)}{x} \qquad \text{for all } x \in (0,b],$$

i.e.

$$\varphi'(x) = a\frac{\varphi(x)}{x}, \quad x \in (0,b],$$

where $a := \dfrac{\eta-\gamma}{\gamma}$. Thus, there exist a $\beta \in \mathbb{R}$ such that $\ln\varphi(x) - \ln x^a = \beta$, $x \in (0, b]$, which means that $\varphi(x) = e^\beta x^a$, $x \in (0, b]$. It suffices to put $k := e^\beta > 0$. Of course $a \neq 0$ (by assumption $\varphi(0) = 0$). It remains to show that $a \in (0, c]$. If we had $a < 0$, then φ would fail to be continuous at zero; thus $a \in (0, \infty)$. Since the function $(0,b] \ni x \mapsto \dfrac{\varphi(x)}{x^c} = kx^{a-c}$ attains its absolute minimum at b we get $a \leq c$. This ends the proof.

THEOREM 3. *Let* $\varphi : [0, b] \to \mathbb{R}$ *be continuous function which is (right-hand side) differentiable at zero. If* $\varphi(0) = 0$, $\varphi(x) > 0$ *for* $x \in (0, b]$ *and if the operator* F^1_φ *given by* (3) *transforms the class* $K(b)$ *into itself, then there exist real constants* $k > 0$ *and* $a > 0$ *such that* $\varphi(x) = kx^a$ *for all* $x \in [0, b]$. *Conversely, if* $\varphi(x) = kx^a$, $x \in [0, b]$, $a, k \in (0, \infty)$, *then the operator* F^1_φ *transforms the class* $K(b)$ *into itself.*

Proof. By assumption, if $f \in K(b)$ then $F^1_\varphi(f)$ is convex. In particular, $g := F^1_\varphi(\mathrm{id})$ is convex on $[0, b]$, positive on $(0, b]$ and $g(0) = 0$. Obviously, the operator F^1_φ, is linear which implies that the function $-g = -F^1_\varphi(\mathrm{id}) = F^1_\varphi(-\mathrm{id})$ is convex as well, because $-\mathrm{id} \in K(b)$. Hence

$$g(\lambda x+(1-\lambda)y)=\lambda g(x)+(1-\lambda)g(y) \quad \text{for all} \quad g(x)\in[0,1],\ x,y\in[0,b].$$

Setting here $y = 0$ and taking into account that $g(0) = 0$, we get

$$g(\lambda x)=\lambda g(x) \quad \text{for} \quad \lambda\in[0,1],\ x\in[0,b].$$

Putting here $x = b$, $t := \lambda b$, $\lambda \in [0, 1]$, we obtain

$$g(t)=\frac{1}{b}\,tg(b)=\left(\frac{1}{b\varphi(b)}\int_0^b\varphi(s)\mathrm{d}s\right)t \quad \text{for all} \quad t\in[0,b],$$

which means that

$$g(t)=\alpha t,\ t\in[0,1],$$

where $\alpha:=\dfrac{1}{b\varphi(b)}\displaystyle\int_0^b\varphi(s)\,\mathrm{d}s>0.$

Consequently

$$\alpha x=\frac{1}{\varphi(x)}\int_0^x\varphi(t)\,\mathrm{d}t \quad \text{for} \quad x\in(0,b],$$

i.e.

(6) $$\varphi(x)=\frac{1}{\alpha x}\int_0^x\varphi(t)\,\mathrm{d}t \quad \text{for} \quad x\in(0,b];$$

in particular, the function $\varphi\,|_{(0,\,b]}$ is differentiable.

From (6) we get

$$\frac{\varphi'(x)}{\varphi(x)}=\frac{1-\alpha}{\alpha}\frac{1}{x},\ x\in(0,b],$$

which implies

$$\varphi(x)=kx^a,\ \ x\in(0,b],$$

with $a:=\dfrac{1-\alpha}{\alpha}$ and $k > 0$. We have $a > 0$ in view of the fact that φ is continuous at zero (by assumption $\varphi(0) = 0$). The other part of the assertion was already proved in [1]. This completes the proof.

Remark 1. As a matter of fact, the assumption that the operator $F_{\varphi,\psi}$ transforms the class $K(b)$ into itself, occurring in Theorems 1 and 2, was used exclusively in order to prove that the functions $F_{\varphi,\psi}(\mathrm{id})$, $F_{\varphi,\psi}(-\mathrm{id})$ are starshaped.

Remark 2. Assumption (M) occurring in the statement of Theorem 2 is trivially satisfied whenever the function

$$(0,b]\ni x\mapsto\frac{\varphi(x)}{x^c}\in\mathbb{R}$$

is monotonically (weakly) decreasing.

Bearing Remarks 1 and 2 in mind and applying Theorem 2 with $c = 2$ we obtain immediately the following

THEOREM 4. *Let $\varphi,\psi:[0,b]\to\mathbb{R}$, be continuous functions such that $\varphi(0)=0$, $\varphi(x)>0$ and $\psi(0)>0$ for $x\in(0,b]$ and let φ be (right-hand side) differentiable at zero. Assume that ψ is starshaped and the function $(0,b]\ni x\mapsto\frac{1}{x^2}\varphi(x)\in\mathbb{R}$ is (weakly) decreasing. If the operator $F_{\varphi,\psi}$ given by* (2) *has the property that the functions $F_{\varphi,\psi}(\mathrm{id})$, $F_{\varphi,\psi}(-\mathrm{id})$ are starshaped, then there exists a real constant $k>0$ and an $a\in(0,2)$ such that*

$$\varphi(x)=kx^a,\ \ \psi(x)=\varphi'(x)=akx^{a-1},\ \ x\in[0,b].$$

REFERENCE

1. Toader, Gh., *On the hierarchy of convexity of functions*, Anal. Numer. Th. Approx., 15 (1986), 167–172.

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