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CONSTRUCTION OF BASKAKOV-TYPE OPERATORS BY WAVELETS

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1. INTRODUCTION

The Baskakov operators are defined on $C[0,\infty)$ as

(1) $$(B_n f)(x)=\sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{k}{n}\right),$$

where

(2) $$b_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}.$$

If we define the mth order central moment by

$$\mu_{n,m}(x)=\sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n}-x\right)^m, \quad m=0,1,2,\ldots,$$

then there are well-known the following relations:

(3) $$\mu_{n,0}(x)=1,\ \mu_{n,1}(x)=0,\ \mu_{n,2}(x)=\frac{x(x+1)}{n}$$

and

$$n\mu_{n,m+1}(x)=x(1+x)\left(\mu'_{n,m}(x)+m\mu_{n,m-1}(x)\right),\quad m=2,3,4,\ldots$$

The purpose of this paper is to introduce a class of Baskakov-type operators by means of Daubechies' compactly-supported wavelets. These new operators have the same moments as Baskakov operators in an arbitrarily chosen number. The rate of convergence of these operators is in connection with the Lipschitz

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functions with respect to the second-order modulus of smoothness. We mention that using wavelets for Szász-type operators, remarkable results have been obtained in [4].

2. PRELIMINARIES

Because the operators defined by (1) cannot be used for L_p-approximation $(1 \le p \le \infty)$, they were modified in an integral extension in the sense of Kantorovich. The Baskakov-Kantorovich operators are given by

(4) $$(B_n^* f)(x) = n\sum_{k=0}^{\infty} b_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(u)du,$$

where $b_{n,k}(x)$ is defined at (2).

There has been an extensive study of relations between rate of convergence and smoothness in $L_p[0,\infty)$.

We will recall the step-weight function φ of operators B_n and B_n^* which controls their rate of convergence. This function is defined as

(5) $$\varphi^2(x) = x(1+x),\ x \ge 0.$$

On the other hand, we recall some facts about wavelets (see [2], [5]). The term "wavelets" refers to sets of functions of the form

$$\psi_{a,b}(x) = a^{-1/2}\psi\left(\frac{x-b}{a}\right),\ a > 0,\ b \in \mathbf{R},$$

i.e., sets of functions formed by dilations and translations of a single function ψ called the "mother wavelet" or "basic wavelet". In the Franklin-Strömberg theory a is replaced by 2^{-j} and b is replaced by $k \cdot 2^{-j}$, where $j, k \in \mathbf{Z}$. In the analysis of an arbitrary function $f \in L_2(\mathbf{R})$, these wavelets are going to play the role of an orthonormal basis. The synthesis of f is given by

$$f(x) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\alpha(j,k)\psi_{j,k}(x),$$

where

$$\alpha(j,k) = 2^{j/2}\int_{\mathbf{R}} f(x)\overline{\psi(2^j x - k)}\,dx.$$

For each positive integer r Ingrid Daubechies constructs an orthonormal basis for $L_2(\mathbf{R})$ of the form

$$2^{j/2}\psi_r(2^j x - k),\ j,k \in \mathbf{Z},$$

where the support of ψ_r is $[0, 2r+1]$. Also, there exists a positive constant γ such that ψ_r has γr continuous derivatives and for any $0 \le k \le r$

$$\int_{\mathbf{R}} x^k \psi_r(x)dx = 0.$$

In the case of Daubechies' wavelets a regular function is approximated by functions that have strong discontinuities.

We mention that for $r=0$ this system reduces to the Haar system. If we want wavelets to be useful for the analysis of other function spaces, it is necessary to impose conditions on them in addition to those we have already given. In what follows we require for $\psi \in L_\infty(\mathbf{R})$ the following conditions:

(C_1) a finite constant $\lambda > 0$ exists with the property $\operatorname{supp}\psi \subset [0,\lambda]$;

(C_2) its first r moments vanish:

$$\int_{\mathbf{R}} t^k\psi(t)dt = 0,\ 1 \le k \le r;$$

(C_3)

$$\int_{\mathbf{R}} \psi(t)dt = 1.$$

Then our Baskakov-type operators are defined as

(6) $$(L_n f)(x) = n\sum_{k=0}^{\infty} b_{n,k}(x)\int_{\mathbf{R}} f(t)\psi(nt-k)dt.$$

If $\chi_{[0,1)}$ is the characteristic function of the interval $[0,1)$ and $\psi = \chi_{[0,1)}$, then L_n becomes B_n^* defined at (4).

In this way, L_n are extensions of the Baskakov-Kantorovich operators. Because $\operatorname{supp}\psi \subset [0,\lambda]$, it is clear that L_n can be written under the following form

(7) $$(L_n f)(x) = \sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\lambda} f\left(\frac{t+k}{n}\right)\psi(t)dt.$$

3. RESULTS

THEOREM 1. *Let L_n be defined by (6). Then, for $0 \le s \le r$, we have*

$$(L_n e_s)(x) = (B_n e_s)(x),\ x \ge 0,$$

where $e_s(t) = t^s$.

Proof. We can write

$$(L_n e_s)(x) = \sum_{k=0}^{\infty} b_{n,k}(x)\int_{\mathbf{R}}\left(\frac{x+k}{n}\right)^s \psi(x)dx = \frac{1}{n^s}\sum_{k=0}^{\infty} b_{n,k}(x)\left(\int_{\mathbf{R}}\sum_{i=0}^{s}\binom{s}{i}x^i k^{s-i}\psi(x)dx\right).$$

By using the conditions (C_1) and (C_2), we obtain the claimed result

$$(L_n e_s)(x)=\frac{1}{n^s}\sum_{k=0}^{\infty} b_{n,k}(x)\int_{\mathbb{R}} k^s\psi(x)\mathrm{d}x=(B_n e_s)(x).$$

The following particular situations will be remarked as important:

(8) $$(L_n e_0)(x) = 1 \text{ and } (L_n e_1)(x) = x.$$

Further, we present some Bernstein-Markov types of inequalities. Firstly, we insert $C_B[0,\infty):=C[0,\infty)\cap L_\infty[0,\infty)$.

THEOREM 2. *Let* $f\in C_B[0,\infty)$. *Then the following inequalities*

(9) $$\|L_n f\|_\infty \le M\|f\|_\infty,$$

(10) $$\|L'_n f\|_\infty \le 2nM\|f\|_\infty,$$

(11) $$\|L''_n f\|_\infty \le 4n(n+1)M\|f\|_\infty$$

hold, where $M=\lambda\|\psi\|_\infty$.

Proof. If we put

$$\int_0^\lambda f\left(\frac{t+k}{n}\right)\psi(t)\mathrm{d}t = I_f(k,n),$$

then, for $f\in C_B[0,\infty)$, the following inequality

$$|I_f(k,n)|\le \lambda\|f\|_\infty\|\psi\|_\infty$$

holds.

By using relation (7), we can write

$$|(L_n f)(x)| \le \sum_{k=0}^{\infty} b_{n,k}(x)\,|I_f(k,n)| \le M\|f\|_\infty.$$

In order to prove relations (10) and (11), we recall

$$b'_{n,k}(x)=\frac{n}{x(1+x)}\left(\frac{k}{n}-x\right)b_{n,k}(x),\ x>0;$$

$$\frac{k}{n}b_{n,k}(x)=xb_{n+1,k-1}(x),\ k=1,2,3,\ldots$$

Thus, we can write

$$(L'_n f)(x) = \frac{n}{1+x}\left\{\sum_{k=1}^{\infty} b_{n+1,k-1}(x)I_f(k,n)-(L_n f)(x)\right\}=$$

(12) $$=\frac{n}{1+x}\left\{\sum_{k=0}^{\infty} b_{n+1,k}(x)I_f(k+1,n)-(L_n f)(x)\right\}.$$

This identity together with relation (9) imply

$$|(L'_n f)(x)| \le \frac{n}{1+x}\left\{\sum_{k=0}^{\infty} b_{n+1,k}(x)|I_f(k+1,n)|+|(L_n f)(x)|\right\} \le 2n\lambda\|f\|_\infty\|\psi\|_\infty.$$

Hence (10) holds.

We derive relation (12) with respect to x and we get

$$(L''_n f)(x)=-\frac{n+1}{1+x}(L'_n f)(x)+\frac{n(n+1)}{(1+x)^2}\left(\sum_{k=1}^{\infty} b_{n+2,k-1}(x)I_f(k+1,n)-\sum_{k=0}^{\infty} b_{n+1,k}(x)I_f(k+1,n)\right).$$

From the above relation and inequality (10) we obtain

$$|(L''_n f)(x)|\le (n+1)|(L'_n f)(x)|+2n(n+1)\lambda\|f\|_\infty\|\psi\|_\infty \le 4n(n+1)M\|f\|_\infty.$$

The proof of this theorem is complete.

THEOREM 3. *Let* $f\in C^1[0,\infty)\cap L_\infty[0,\infty)$. *Then the following inequality*

$$\|L'_n f\|_\infty \le M(\|f\|_\infty+\|f'\|_\infty)$$

holds, where $M=\lambda\|\psi\|_\infty$.

Proof. Relation (14) may be written as follows

$$(L'_n f)(x)=\frac{n}{1+x}\left\{\sum_{k=0}^{\infty} b_{n,k}(x)(I_f(k+1,n)-I_f(k,n))+\sum_{k=0}^{\infty}(b_{n+1,k}(x)-b_{n,k}(x))I_f(k+1,n)\right\}.$$

But we have

$$|I_f(k+1,n)-I_f(k,n)|\le \frac{\lambda}{n}\|f'\|_\infty\|\psi\|_\infty,$$

because

$$f\left(\frac{t+k+1}{n}\right)-f\left(\frac{t+k}{n}\right)=\frac{1}{n}f'(\xi_k),$$

ξ_k lies between $\dfrac{t+k}{n}$ and $\dfrac{t+k+1}{n}$. Also, it is easy to verify that

$$b_{n+1,k}(x) - b_{n,k}(x) = \frac{\frac{k}{n} - x}{1+x}\, b_{n,k}(x).$$

All these relations lead to the following inequality

$$\left|(L_n' f)(x)\right| \le \lambda \|\psi\|_\infty \left(\frac{1}{1+x}\|f'\|_\infty + \frac{n}{(1+x)^2}\|f\|_\infty \sum_{k=0}^{\infty} b_{n,k}(x)\left|\frac{k}{n} - x\right| \right).$$

Applying the Cauchy's inequality, we obtain

$$\sum_{k=0}^{\infty} b_{n,k}(x)\left|\frac{k}{n} - x\right| \le \sum_{k=0}^{\infty} b_{n,k}(x)\left(\frac{k}{n} - x\right)^2 = \mu_{n,2}(x)$$

and, according to (3), we get the desired result.

Further, let $g \in C^2[0,\infty) \cap C_B[0,\infty)$. We use the Taylor expansion

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u) g''(u)\,du.$$

By using relations (8) and (3), for $x > 0$, we can write successively

$$\left|(L_n g)(x) - g(x)\right| = \left| L_n\left(\left(\int_x^{\cdot} (\cdot - u) g''(u)\,du \right), x \right) \right| =$$

$$= \left| \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\lambda} \left(\int_x^{\frac{t+k}{n}} \left(\frac{t+k}{n} - u \right) g''(u)\,du \right) \psi(t)\,dt \right| \le$$

$$\le \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\lambda} \left(\left| \int_x^{\frac{t+k}{n}} \left| \frac{t+k}{n} - u \right| |g''(u)|\,du \right| \right) dt \|\psi\|_\infty \le$$

$$\le \|\psi\|_\infty \|g''\|_\infty \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\lambda} \left(\frac{t+k}{n} - x \right)^2 dt =$$

$$= \lambda \|\psi\|_\infty \|g''\|_\infty \left(\frac{\lambda^2}{3n^2} + \frac{\lambda}{n}\mu_{n,1} + \mu_{n,2}(x) \right) = M\|g''\|_\infty \left(\frac{\lambda^2}{3n^2} + \frac{x(x+1)}{n} \right).$$

Taking the infimum over $g \in C^2[0,\infty) \cap C_B[0,\infty)$, by (9), we have

$$\left|(L_n f)(x) - f(x)\right| \le \inf_{g \in C^2 \cap C_B} \left\{ \|L_n(f-g)\|_\infty + \|f-g\|_\infty + \left|(L_n g)(x) - g(x)\right| \right\} \le$$

$$\le \inf_{g \in C^2 \cap C_B} \left\{ (M+1)\|f-g\|_\infty + M\|g''\|_\infty \left(\frac{\lambda^2}{3n^2} + \frac{x(x+1)}{n} \right) \right\} \le$$

$$\le (M+1) \inf_{g \in C^2 \cap C_B} \left\{ \|f-g\|_\infty + \left(\frac{\lambda^2}{3n^2} + \frac{x(x+1)}{n} \right) \|g''\|_\infty \right\} =$$

$$= (M+1) K_2\left(f, \frac{\lambda^2}{3n^2} + \frac{x(x+1)}{n} \right).$$

Here, K_2 is Peetre's functional given by

$$(13) \qquad K_2(f,t) = \inf_{g \in C^2 \cap C_B} \left\{ \|f-g\|_\infty + t\|g''\|_\infty \right\},\ t > 0.$$

Now, we can state the following result

THEOREM 4. *Let $f \in C_B[0,\infty)$ and L_n be given by (6). Then*

$$\left|(L_n f)(x) - f(x)\right| \le (M+1) K_2\left(f, \frac{\varphi^2(x)}{n} + \frac{a}{n^2} \right),$$

where $M = \lambda\|\psi\|_\infty$, $a = \dfrac{\lambda^2}{3}$, φ and K_2 are mentioned at (5), respectively at (13).

It is known that Peetre's functional is equivalent to the regular modulus of smoothness, consequently there exist some constants $\beta > 0$ and $t_0 > 0$ such as

$$(14) \qquad \beta^{-1}\omega_2(f,t) \le K_2(f,t^2) \le \beta\omega_2(f,t),\ f \in C_B,\ 0 < t \le t_0,$$

where

$$\omega_2(f,t) = \sup_{0<h\le t} \|\Delta_h^2 f\|_\infty$$

and

$$\Delta_h^2 f(x) = \begin{cases} f(x+h) - 2f(x) + f(x-h), & \text{when } h \le x \\ 0, & \text{otherwise.} \end{cases}$$

Once Theorem 4 and inequality (14) are known, we can easily obtain

COROLLARY. *Let $0 < \alpha < 2$, $f \in C_B[0,\infty)$ and L_n given at (6). If*

$$\omega_2(f,t) = O(t^\alpha),$$

then we have

$$|(L_nf)(x) - f(x)| \le \gamma\left(\frac{\varphi^2(x)}{n} + \frac{a}{n^2}\right)^{\frac{\alpha}{2}},$$

where $a = \frac{\lambda^2}{3}$, γ *is a constant and* φ *is defined by* (5).

Next, let s be an integer such that $\lambda \le s$ and $f \in L_1[0,\infty)$. We can write

$$\|L_nf\|_1 \le \int_0^\infty \sum_{k=0}^\infty \left(\int_0^\lambda \left|f\left(\frac{t+k}{n}\right)\right| \|\psi\|_\infty \mathrm{d}t\right) b_{n,k}(x)\mathrm{d}x \le$$

$$\le \|\psi\|_\infty \sum_{k=0}^\infty \int_0^\lambda \left|f\left(\frac{t+k}{n}\right)\right| \mathrm{d}t \int_0^\infty b_{n,k}(x)\mathrm{d}x = \frac{n}{n-1}\|\psi\|_\infty \sum_{k=0}^\infty \int_{\frac{k}{n}}^{\frac{\lambda+k}{n}} |f(x)|\mathrm{d}x.$$

Here we have used the fact that for $n \ge 2$

$$\int_0^\infty b_{n,k}(x)\mathrm{d}x = \frac{1}{n-1}.$$

On the other hand,

$$\sum_{k=0}^\infty \int_{\frac{k}{n}}^{\frac{\lambda+k}{n}} |f(x)|\mathrm{d}x \le \sum_{k=0}^\infty \int_{\frac{k}{n}}^{\frac{s+k}{n}} |f(x)|\mathrm{d}x = \sum_{k=0}^\infty \sum_{i=0}^{s-1} \int_{\frac{k+i}{n}}^{\frac{k+i+1}{n}} |f(x)|\mathrm{d}x \le \sum_{i=0}^{s-1} \|f\|_1 = s\|f\|_1.$$

Hence it follows that $\|L_nf\|_1 \le M_1\|f\|_1$, where $M_1 = \frac{sn}{n-1}\|\psi\|_\infty$.

The above inequality and relation (9), by means of Riesz-Thorin theorem (see [1]), lead to the following result

THEOREM 5. *Let* $n>1$, $1 \le p \le \infty$, $f \in L_p[0,\infty)$. *Then*

$$\|L_nf\|_p \le C_p\|f\|_p$$

holds, where C_p *is a constant.*

The Riesz-Thorin theorem claims that the norm of operator L_n does not exceed $M_1^{1-\nu}M^\nu$, where, in our case, $p^{-1} = 1-\nu$, $\nu \in (0,1)$. After a few calculations we obtain an upper bound of the operator's norm, as follows

$$C_p = \lambda^{1-\frac{1}{p}} s^{\frac{1}{p}} \left(\frac{n}{n-1}\right)^{\frac{1}{p}} \|\psi\|_\infty.$$

For the classical operator B_n^* the estimation

$$\|B_n^*f\|_p \le \left(\frac{n}{n-1}\right)^{\frac{1}{p}} \|f\|_p \tag{15}$$

is known (see [3], p. 118).

If we choose $\psi = \chi_{[0,1)}$, then $\|\psi\|_\infty = 1$, $\lambda = 1$ and s can become λ. In this way, we come across (15).

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