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SEQUENCES OF MULTICRITERIA OPTIMIZATION PROBLEMS

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A whole series of production processes, economic systems of different types and technical objectives are described by mathematical models which are multicriteria optimization problems.

This paper presents some properties of different classes of multicriteria optimization problem solutions.

Consider the following model of a multicriteria optimization problem

$$(P)\begin{cases} \text{v}-\min f(x) \\ x \in X, \end{cases}$$

where X is a nonempty convex compact set of $\Re^n$ and $f = (f_1,\ldots,f_m): X \to \Re^m$ is a continuous function on X.

Let us recall some concepts of multicriteria optimization problem solutions:

DEFINITION 1. The point $x^P \in X$ is said to be a Pareto solution of Problem (P) if there exists no point $x \in X$ such that $f(x) \leq f(x^P)$.

The inequality $f(x) \leq f(x^P)$ means

$$f_i(x) \leqq f_i(x^P) \text{ for all } i \in \{1,\ldots,m\}$$

and

$$f_1(x)+\ldots+f_m(x) < f_1(x^P)+\ldots+f_m(x^P).$$

Let $P(f; X)$ denote the set of Pareto solutions for Problem (P).

DEFINITION 2. The point $x^S \in X$ is said to be a Slater solution of Problem (P) if there exists no point $x \in X$ such that

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$$f_i(x) < f_i(x^S) \text{ for all } i \in \{1,\ldots,m\}.$$

Let $S(f; X)$ denote the set of Slater solutions for Problem (P).

DEFINITION 3. The point $x^G \in X$ is said to be a Geoffrion solution of Problem (P) if x^G is a Pareto solution of Problem (P) and there exists a positive number $M > 0$ such that for each $i \in \{1,\ldots,m\}$ we have

$$\frac{f_i(x^G) - f_i(x)}{f_j(x) - f_j(x^G)} \leq M,$$

for some $j \in \{1,\ldots,m\}$ such that $f_j(x^G) < f_j(x)$ whenever $x \in X$ and $f_i(x) < f_i(x^G)$.

Let $G(f; X)$ denote the set of Geoffrion solutions for Problem (P).

It is obvious that

$$G(f;X) \subseteq P(f;X) \subseteq S(f;X).$$

If $A = [a_{ij}] \in \Re^{p\times m}$ is a matrix of the type (p, m) with all real elements, then $Af : X \to \Re^p$ denotes the function defined by

$$Af(x) = (a_{11}f_1(x)+\ldots+a_{1m}f_m(x), \ldots, a_{p1}f_1(x)+\ldots+a_{pm}f_m(x))$$

for all $x \in X$.

THEOREM 1. *Let $A = [a_{ij}] \in \Re^{p\times m}$ be a matrix such that $(a_{i1},\ldots,a_{im}) \geq 0$ for all $i \in \{1,\ldots,p\}$. If x^S is a Slater solution of Problem*

$$(PA)\begin{cases} V - \min Af(x) \\ x \in X, \end{cases}$$

then x^S is a Slater solution of Problem (P).

Proof. Assume that x^S is not a Slater solution of Problem (P); then there exists a point $x \in X$ such that $f(x) < f(x^S)$. From this, because $(a_{i1},\ldots,a_{im}) \geq 0$ for all $i \in \{1,\ldots,p\}$, it follows that

$$a_{i1}f_1(x)+\ldots+a_{im}f_m(x) < a_{i1}f_1(x^S)+\ldots+a_{im}f_m(x^S)$$

for all $i \in \{1,\ldots,p\}$, hence $Af(x) < Af(x^S)$. Therefore, x^S is not a Slater solution of Problem (PA), which is a contradiction.

Remark 1. The converse of Theorem 1 is not always true. For example, let

$$X = \{(x_1,x_2) \in \Re^2 \mid 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\},$$

$$f(x_1,x_2) = (x_1,x_2) \text{ for all } (x_1,x_2) \in X$$

and

$$A = [11] \in \Re^{1\times 2}.$$

Then

$$S(f;X) = \{(x_1,x_2) \in X | x_1 = 0, x_2 \in [0,1]\} \cup \{(x_1,x_2) \in X | x_2 = 0, x_1 \in [0,1]\}$$

and

$$S(Af;X) = \{(0,0)\}.$$

Hence, the point (1, 0) is a Slater solution of Problem (P), but it is not a Slater solution of Problem (PA).

THEOREM 2. *Let $A = [a_{ij}] \in \Re^{p\times m}$ be a matrix such that $a_{ij} > 0$ for all $i \in \{1,\ldots,p\}$ and $j \in \{1,\ldots,m\}$. If x^P is a Pareto solution of Problem (PA), then x^P is a Pareto solution of Problem (P).*

Proof. Assume that x^P is not a Pareto solution of Problem (P); then there exists a point $x \in X$ such that $f(x) \leq f(x^P)$. From this, because $a_{ij} > 0$ for all $i \in \{1,\ldots,p\}$ and for all $j \in \{1,\ldots,m\}$, it follows that

$$(a_{11}f_1(x)+\ldots+a_{1m}f_m(x), \ldots, a_{p1}f_1(x)+\ldots+a_{pm}f_m(x)) \leq$$
$$\leq (a_{11}f_1(x^P)+\ldots+a_{1m}f_m(x^P), \ldots, a_{p1}f_1(x^P)+\ldots+a_{pm}f_m(x^P)),$$

hence $Af(x) \leq Af(x^P)$. Therefore, x^P is not a Pareto solution of Problem (PA), which is a contradiction.

Remark 2. The converse of Theorem 2 is not always true. For example, let

$$X = \{(x_1,x_2) \in \Re^2 | x_1 \in [0,1], x_2 \in [0,1], x_1 + x_2 \geq 1\},$$

$$f(x_1,x_2) = (x_1,x_2) \text{ for all } (x_1,x_2) \in X,$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \in \Re^{2\times 2}.$$

Then

$$P(f;X) = \{(x_1, x_2) \in X | x_1 + x_2 = 1\}$$

and

$$P(Af;X) = \{(1,0)\}.$$

Hence, the point (0, 1) is a Pareto solution of Problem (*P*), but it is not a Pareto solution of Problem (*PA*).

THEOREM 3. *Let* $A = [a_{ij}] \in \Re^{p\times m}$ *be a matrix such that* $a_{ij} > 0$ *for all* $i \in \{1,\dots,p\}$ *and* $j \in \{1,\dots,m\}$. *If* x^S *is a Slater solution of Problem (PA), then* x^S *is a Geoffrion solution of Problem (P).*

Proof. Since x^S is a Slater solution of Problem (*PA*), there exists no point $x \in X$ such that $Af(x) < Af(x^S)$. This means that, for each $x \in X$, there exists an index $k = k(x) \in \{1,\dots,p\}$ such that the inequality

$$(1) \qquad a_{k1}f_1(x)+\dots+a_{km}f_m(x) \geq a_{k1}f_1(x^S)+\dots+a_{km}f_m(x^S)$$

is fulfilled. Let $c_k = a_{k1}+\dots+a_{km}$. From $a_{kj} > 0$ for all $j \in \{1,\dots,m\}$ it follows that $c_k > 0$ and hence by (1) we deduce

$$\sum_{j=1}^{m}\frac{a_{kj}}{c_k}f_j(x) \geq \sum_{j=1}^{m}\frac{a_{kj}}{c_k}f_j(x^S).$$

Let now $c_i = a_{i1}+\dots+a_{im}$ for all $i \in \{1,\dots,p\}$. Obviously, $c_i > 0$ for all $i \in \{1,\dots,p\}$. Then there exist the points

$$v^i = \left(\frac{a_{i1}}{c_i},\dots,\frac{a_{im}}{c_i}\right), \quad i \in \{1,\dots,p\},$$

such that $v^i = (v_1^i,\dots,v_m^i) > 0$ for all $i \in \{1,\dots,p\}$ and, for each $x \in X$, there exists an index $k = k(x) \in \{1,\dots,p\}$ so that

$$\sum_{j=1}^{m} v_j^k f_j(x) \geq \sum_{j=1}^{m} v_j^k f_j(x^S).$$

Now, in view of Theorem 2.1.12 in [2], the point x^S is a Geoffrion solution of Problem (*P*).

Let now $B^1 \in \Re^{p_1\times m}, B^2 \in \Re^{p_2\times p_1},\dots$ be a sequence of matrices and let $(A^k)_{k\in\mathbf{N}}$ be the sequence of matrices defined by

$$(2) \qquad \begin{cases} A^1 = B^1 \\ A^k = B^k A^{k-1}, \ k \in \mathbf{N}, k \geq 2. \end{cases}$$

Assume that each matrix $A^k = [a_{ij}^k], (k \in \mathbf{N})$ has all positive elements. Consider the family of multicriteria optimization problems

$$(PA^k) \begin{cases} \text{v} - \min A^k f(x) \\ x \in X, \end{cases}$$

where $k \in \mathbf{N}$.

Such problems chains usually occur in sequential processes of taking decisions; at each step the decision is taken considering the state of the system; the utility functions at each step have a ratio corresponding to the priority of the moment in which the decision is taken.

From Theorems 1, 2 and 3 it follows

THEOREM 4. *The chain of inclusions*

$$S(f;X) \supseteq P(f;X) \supseteq G(f;X) \supseteq S(A^1f;X) \supseteq P(A^1f;X) \supseteq G(A^1f;X) \supseteq \dots$$

$$\dots \supseteq S(A^kf;X) \supseteq P(A^kf;X) \supseteq G(A^kf;X) \supseteq \dots$$

is valid.

If $X \subseteq \Re^n$ is a nonempty compact set and the function $f: X \to \Re^n$ is continuous, then the set $S(f;X)$ is a nonempty compact set. Hence each set $S(A^kf;X), (k \in \mathbf{N})$ is a nonempty compact set. Then, by a well-known theorem of functional analysis, the family of nested sets $(S(A^kf;X))_{k\in\mathbf{N}}$ has a nonempty intersection; this means that there exists a set $X^* \subseteq X$ such that

$$X^* = \bigcap\{S(A^kf;X) | k \in \mathbf{N}\} = \lim_{k\to\infty} S(A^kf;X).$$

Now, assume that for every $k \in \mathbf{N}$ the matrix A^k is square, of the *m*th order, with all positive elements, and there exists

$$\lim_{k\to\infty} A^k = A^* \in \Re^{m\times m}.$$

Is then the equality $X^* = S(A^*f;X)$ true? Or, generally speaking, are the equalities

$$(3) \qquad \lim_{k\to\infty} S(A^kf;X) = S\left(\left(\lim_{k\to\infty} A^k\right)f;X\right),$$

(4) $$\lim_{k\to\infty} P\left(A^k f; X\right) = P\left(\left(\lim_{k\to\infty} A^k\right) f; X\right),$$

(5) $$\lim_{k\to\infty} G\left(A^k f; X\right) = G\left(\left(\lim_{k\to\infty} A^k\right) f; X\right),$$

true?

Example 1. Let us consider the multicriteria optimization problem

$$\begin{cases} v - \min f(x_1, x_2) = (x_1, x_2) \\ (x_1, x_2) \in \left\{(x_1, x_2) \in \Re^2 \middle| x_1 \in [0,1], x_2 \in [0,1], 2x_1 + 2x_2 \geq 1\right\}, \end{cases}$$

and the matrix

(6) $$B^k = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix} \in \Re^{2\times 2}$$

for all $k \in \mathrm{N}$. Then

(7) $$A^k = \begin{pmatrix} \frac{1}{3} + \frac{2}{3}\left(\frac{2}{5}\right)^k & \frac{2}{3} - \frac{2}{3}\left(\frac{2}{5}\right)^k \\ \frac{1}{3} - \frac{1}{3}\left(\frac{2}{5}\right)^k & \frac{2}{3} + \frac{1}{3}\left(\frac{2}{5}\right)^k \end{pmatrix} \in \Re^{2\times 2}$$

for all $k \in \mathrm{N}$; hence, there exists

(8) $$A^* = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \in \Re^{2\times 2}.$$

On the other hand, we have

$$S(f; X) = \left\{(x_1, x_2) \in \Re^2 \middle| 2x_1 + 2x_2 = 1, x_1 \geq 0, x_2 \geq 0\right\} \cup$$

$$\cup \left\{(x_1, x_2) \in \Re^2 \middle| x_1 = 0, x_2 \in \left[\frac{1}{2}, 1\right]\right\} \cup \left\{(x_1, x_2) \in \Re^2 \middle| x_2 = 0, x_1 \in \left[\frac{1}{2}, 1\right]\right\};$$

$$P(f; X) = \left\{(x_1, x_2) \in \Re^2 \middle| 2x_1 + 2x_2 = 1, x_1 \in [0,1], x_2 \in [0,1]\right\};$$

$$G(f; X) = \left\{(x_1, x_2) \in \Re^2 \middle| 2x_1 + 2x_2 = 1, x_1 \in [0,1], x_2 \in [0,1]\right\};$$

$$S\left(A^1 f; X\right) = G\left(A^1 f; X\right) = P\left(A^1 f; X\right) = P(f; X);$$

$$S\left(A^k f; X\right) = G\left(A^k f; X\right) = P\left(A^k f; X\right) = \left\{\left(\frac{1}{2}, 0\right)\right\},$$

for all $k \in \mathrm{N}$ $k \geq 2$. Therefore, there exist

$$\lim_{k\to\infty} S\left(A^k f; X\right) = \lim_{k\to\infty} G\left(A^k f; X\right) = \lim_{k\to\infty} P\left(A^k f; X\right) = \left\{\left(\frac{1}{2}, 0\right)\right\}.$$

It can be easily verified that

$$P\left(A^* f; X\right) = G\left(A^* f; X\right) = S\left(A^* f; X\right) = \left\{\left(\frac{1}{2}, 0\right)\right\};$$

thus, in this case, equalities (3), (4) and (5) are fulfilled.

Example 2. Let us consider the multicriteria optimization problem

$$\begin{cases} v - \min f(x_1, x_2) = (x_1, x_2) \\ (x_1, x_2) \in \left\{(x_1, x_2) \in \Re^2 \middle| x_1 \in [0,1], x_2 \in \left[0, \frac{1}{2}\right], x_1 + 2x_2 \geq 1\right\}, \end{cases}$$

and $B^k \in \Re^{2\times 2}$ $(k \in \mathrm{N})$ the matrix given by (6). Then $A^k \in \Re^{2\times 2}$ $(k \in \mathrm{N})$ is given by (7) and hence A^* is given by (8).

We have

$$S(f; X) = G(f; X) = P(f; X) = S\left(A^k f; X\right) = G\left(A^k f; X\right) = P\left(A^k f; X\right) =$$

$$= \left\{(x_1, x_2) \in \Re^2 \middle| x_1 + 2x_2 = 1, x_1 \in [0,1], x_2 \in [0,1]\right\}$$

for all $k \in \mathrm{N}$ and hence equalities (3), (4) and (5) are fulfilled, too.

Example 3. Let us consider the multicriteria optimization problem

$$\begin{cases} v - \min f(x_1, x_2) = (x_1, x_2) \\ (x_1, x_2) \in \left\{(x_1, x_2) \in \Re^2 \middle| x_1 \in [0,1], x_2 \in [0,1]\right\}, \end{cases}$$

and the matrix

$$B^k = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix} \in \Re^{2\times 2}$$

for all $k \in \mathbf{N}$. Then

$$A^k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \quad \text{for all} \quad k \in \mathbf{N},$$

where

$$a_k = \frac{10+\sqrt{10}}{20}\cdot\left(\frac{5+\sqrt{10}}{12}\right)^k + \frac{10-\sqrt{10}}{20}\cdot\left(\frac{5-\sqrt{10}}{12}\right)^k,$$

$$b_k = c_k = \frac{3\sqrt{10}}{20}\cdot\left(\frac{5+\sqrt{10}}{12}\right)^k - \frac{3\sqrt{10}}{20}\cdot\left(\frac{5-\sqrt{10}}{12}\right)^k,$$

$$d_k = \frac{8+\sqrt{10}}{48}\cdot\left(\frac{5+\sqrt{10}}{12}\right)^{k-1} + \frac{8-\sqrt{10}}{48}\cdot\left(\frac{5-\sqrt{10}}{12}\right)^{k-1}$$

for every $k \in \mathbf{N}$. Therefore, there exists

$$A^* = \lim_{k\to\infty} A^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Obviously,

$$S(A^*f;X) = P(A^*f;X) = G(A^*f;X) = X.$$

On the other hand, we have

$$S(f;X) = P(f;X) = G(f;X) = S(A^kf;X) = P(A^kf;X) = G(A^kf;X) = \{(0,0)\},$$

for all $k \in \mathbf{N}$. Hence there exist

$$\lim_{k\to\infty} S(A^kf;X) = \lim_{k\to\infty} P(A^kf;X) = \lim_{k\to\infty} G(A^kf;X) = \{(0,0)\}.$$

Thus, in this case, equalities (3), (4) and (5) are not fulfilled.

The following theorem shows a relationship among the sets X^* and $S(A^*f;\ X)$.

THEOREM 5. *Let $(B^k)_{k\in\mathbf{N}}$ be a sequence of square matrices of the mth order such that the sequence $(A^k)_{k\in\mathbf{N}}$ defined by (2) has the property that each matrix $A^k = [a_{ij}^k], (k \in \mathbf{N})$, has all positive elements:*

$$a_{ij}^k > 0 \ \textit{for every}\ i,j \in \{1,\dots,m\}\ \textit{and}\ k \in \mathbf{N}.$$

If there exists

$$\lim_{k\to\infty} A^k = A^*,$$

then

(9) $$X^* \subseteq S(A^*f;X).$$

Proof. Let $x \in X^*$. Then there exists a sequence $(x^k)_{k\in\mathbf{N}}$ of points $x^k \in S(A^kf;X)$, $(k \in \mathbf{N})$ such that x^* is the limit of the sequence $(x^k)_{k\in\mathbf{N}}$. From $x^k \in S(A^kf;X)$, $(k \in \mathbf{N})$ we deduce that it does not exist $x \in X$ such that $A^kf(x) < A^kf(x^k)$. This means that, for every $x \in X$, there exists an index $i = i(x) \in \{1,\dots,m\}$ such that

$$\left(A^kf(x) - A^kf(x^k)\right)_i \ge 0.$$

Let $x \in X$ and let

$$K_x^j = \left\{k \in \mathbf{N} \,\middle|\, \left(A^kf(x) - A^kf(x^k)\right)_j \ge 0\right\},\ j \in \{1,\dots,m\}.$$

It is clear that

$$K_x^1 \cup \dots \cup K_x^m = \mathbf{N}$$

and it is obvious that at least one of the sets $K_x^1,\dots,K_x^m$ is infinite; suppose, for instance, that the set K_x^1 is infinite. Since

$$A^* = \lim_{k\to\infty} A^k,$$

we have

$$\lim_{\substack{k\to\infty \\ k\in K_x^1}} A^k = A^*,$$

and

(10) $$\lim_{\substack{k\to\infty \\ k\in K_x^1}} \left(A^kf(x) - A^kf(x^k)\right)_1 = \left(A^*f(x) - A^*f(x^*)\right)_1.$$

Then, from (10) and from the structure of K_x^1, we obtain

$$\left(A^*f(x) - A^*f(x^*)\right)_1 \ge 0.$$

Since x is an arbitrary point in the set X, it follows that, for each $x \in X$, there exists an index $i = i(x) \in \{1, \ldots, m\}$ such that $\left(A^* f(x) - A^* f(x^*)\right)_i \geqq 0$. This means that $x^* \in S\left(A^* f; X\right)$, hence relation (9) is fulfilled.

Remark 3. Considering $n = m = 2$ and the particular case when A is a stochastic matrix, M. E. Salukvadze and A. L. Topchishvili [4] have proved that relation (9) is an equality.

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