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ON A REPRESENTATION OF THE EXPANSION COEFFICIENTS OF A FUNCTION RELATIVE TO A POWER SCALE

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The main results of this paper are: a necessary and sufficient condition for the existence of the asymptotic expansion

$$(1)\qquad f(x_0+h)=\sum_{k=0}^{n}A_kh^k+o\left(h^n\right)$$

of the function f, where o is the Landau symbol, and the representation of the coefficient A_k, $k=1,2,\ldots,n$ as an iterated limit of a difference quotient of f. The number $k!\,A_k$ is a generalization of the kth-order derivative of f at x_0.

Let $I\subset\mathbb{R}$ be an open interval, $x_0\in I$ and let f be a function defined in I that is continuous at x_0. If the variable h belongs to a sufficiently small neighbourhood $U(0)$ of the origin, then $x_0+h\in I$ and we can consider the function $h\mapsto f(x_0+h)$, defined in $U(0)$. The nth-order asymptotic expansion of this function relative to the power scale $\left\{h^k,\ k=0,1,\ldots,n\right\}$ is (1), where the coefficients A_k are calculated successively

$$(2)\qquad A_0=f(x_0),\quad A_k=\lim_{h\to 0}\frac{1}{h^k}\left(f(x_0+h)-\sum_{i=0}^{k-1}A_ih^i\right),\quad k=1,2,\ldots,n,$$

provided that the limits of the second member of (2) exist and are finite.

For $n=1$, the existence of the expansion (1) is equivalent to the derivativeness of f at x_0 and $A_k=f'(x_0)$. The n time derivativeness of f at x_0 is a sufficient condition for the existence of the expansion (1) $\left(\text{where } A_k=\dfrac{1}{k!}f^{(k)}(x_0)\right)$, but for $n\geq 2$ this condition is not a necessary one. For $n=2$ and $x_0=0$, the following example may be an illustration of this fact

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$$(3)\qquad f(x)=\begin{cases} x^3\sin\dfrac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0.\end{cases}$$

To obtain a condition which is also a necessary one, for each natural number k, $1\le k\le n$, the difference quotient

$$(4)\qquad [x_0, x_0+h_1,\ldots,x_0+h_k; f]$$

is considered, where the variables h_i, $i=1,2,\ldots,k$ satisfy $h_i\in U(0)$ and for $i\neq j$, $h_i\neq h_j$.

LEMMA. *If* $[x_0; f]=f(x_0)=A_0$ *and for each natural* $k, 1\le k\le m$,

$$(5)\qquad \lim_{h_k\to 0,\ldots,h_1\to 0}[x_0, x_0+h_1,\ldots,x_0+h_k; f]=A_k,\ A_k\in\mathbf{R},$$

then

$$(6)\qquad \lim_{h_m\to 0,\ldots,h_1\to 0}[x_0,x_0+h_1,\ldots,x_0+h_m,x_0+h_{m+1}; f]=\frac{f(x_0+h_{m+1})-\sum_{k=0}^{m}A_k h_{m+1}^k}{h_{m+1}^{m+1}}.$$

Proof. For $m=0$, the statement is obvious. We suppose that it is true for each natural number less than m and we will show that it is true for m, i.e., that relation (6) takes place. The left-hand side of (6) is equal to

$$\lim_{h_m\to 0,\ldots,h_1\to 0}\frac{1}{h_{m+1}}\left([x_0+h_1,\ldots,x_0+h_{m+1}; f]-[x_0,x_0+h_1,\ldots,x_0+h_m; f]\right)=$$

$$=\frac{1}{h_{m+1}}\left(\lim_{h_m\to 0,\ldots,h_2\to 0}[x_0,x_0+h_2,\ldots,x_0+h_{m+1}; f]-A_m\right).$$

Since the statement is supposed to be true for $m-1$, the equality

$$\lim_{h_m\to 0,\ldots,h_2\to 0}[x_0,x_0+h_2,\ldots,x_0+h_{m+1}; f]=\frac{f(x_0+h_{m+1})-\sum_{k=0}^{m-1}A_k h_{m+1}^k}{h_{m+1}^m}$$

takes place. Using this equality, the left-hand side of (6) becomes equal to

$$\frac{1}{h_{m+1}}\left(\frac{f(x_0+h_{m+1})-\sum_{k=0}^{m-1}A_k h_{m+1}^k}{h_{m+1}^m}-A_m\right),$$

which is equal to the right-hand side of (6).

THEOREM. *The necessary and sufficient condition for the existence of the expansion* (1) *is that for each natural number* $k, 1\le k\le n$, *the difference quotient* (4) *has a finite iterated limit, when* $h_1\to 0,\ldots,h_k\to 0$. *In this case, the coefficients* A_k, $k=1,2,\ldots,n$, *are*

$$(7)\qquad A_k=\lim_{h_k\to 0,\ldots,h_1\to 0}[x_0,x_0+h_1,\ldots,x_0+h_k; f].$$

Proof. Relation (1) is equivalent to relations (2).

If we prove that for each $k, 1\le k\le n$, relations

$$(8)\qquad \lim_{h\to 0}\frac{f(x_0+h)-\sum_{i=0}^{k-1}A_i h^i}{h^k}=\lim_{h_k\to 0,\ldots,h_1\to 0}[x_0,x_0+h_1,\ldots,x_0+h_k; f]$$

take place, then both the necessity and sufficiency of the condition result immediately.

We will prove (8) by induction. For $k=1$, it can be easily verified. Let m be a natural number such that $1\le m\le n-1$ and we suppose that (8) is true for each natural $k, k\le m$. We will show that (8) is true for $m+1$. The following equalities take place

$$\lim_{h_{m+1}\to 0,\ldots,h_1\to 0}[x_0,x_0+h_1,\ldots,x_0+h_{m+1}; f]=$$

$$=\lim_{h_{m+1}\to 0}\frac{1}{h_{m+1}}\lim_{h_m\to 0,\ldots,h_1\to 0}\left([x_0+h_1,x_0+h_2,\ldots,x_0+h_{m+1}; f]-[x_0,x_0+h_1,\ldots,x_0+h_m; f]\right)=$$

$$=\lim_{h_{m+1}\to 0}\frac{1}{h_{m+1}}\left(\lim_{h_m\to 0,\ldots,h_2\to 0}[x_0,x_0+h_2,\ldots,x_0+h_{m+1}; f]-A_m\right)=$$

$$=\lim_{h_{m+1}\to 0}\frac{f(x_0+h_{m+1})-\sum_{k=0}^{m}A_k h_{m+1}^k}{h_{m+1}^{m+1}}$$

(note that the last equality is true since, according to the above lemma,

$$\lim_{h_m \to 0,\dots,h_2\to 0}[x_0, x_0+h_2,\dots,x_0+h_{m+1};f] = \frac{f(x_0+h_{m+1})-\sum_{k=0}^{m-1}A_k h_{m+1}^k}{h_{m+1}^m}).$$

Replacing in the last expression h_{m+1} by h, we obtain

$$\lim_{h_{m+1}\to 0,\dots,h_1\to 0}[x_0, x_0+h_1,\dots,x_0+h_{m+1};f] = \lim_{h\to 0}\frac{f(x_0+h)-\sum_{k=0}^{m}A_k h^k}{h^{m+1}},$$

which shows that relation (8) is true for $m+1$.

Remark 1. For $k \geq 2$, the number $k!A_k$, where A_k is given by (7), is a generalization of the kth-order derivative of f at x_0. We emphasize that this notion is different from that based on the limit of difference quotient, when the variables $h_1,\dots, h_k$ tend simultaneously to 0, i.e., on the limit of function

$$(h_1,\dots,h_k)\mapsto[x_0, x_0+h_1,\dots,x_0+h_k;f]$$

at the point $O(0,\dots,0)\in\mathbf{R}^k$. This generalization also differs from the so-called direct derivative of the order k of f at x_0. We will illustrate these, for $k=2$, by the function (3), at the point $x_0=0$. It is easy to verify that

$$\lim_{h''\to 0, h'\to 0}[0,h',h'';f] = \lim_{h''\to 0} h''\sin\frac{1}{h''} = 0.$$

To show that the function $(h',h'')\mapsto[0,h',h'';f]$ has not a limit at $O(0,0)$, we consider the sequences $h'_n = \frac{1}{n\pi}$, $h''_n = \frac{2}{(2n-1)\pi}$, $n = 1, 2, \dots$. The sequence $[0,h'_n,h''_n;f]$ has not a limit, when $n\to\infty$. The direct derivative of the second order of f at the origin $O(0,0)$ is the limit at O of the function $(h',h'')\mapsto Q(h',h'')$, where

$$Q(h',h'') = \frac{f(h'+h'')-f(h'')-f(h')+f(0)}{h'h''}.$$

To show that $Q(h',h'')$ has not a limit at O, we consider the sequences

$$h'_n = \frac{1}{n\pi},\ h''_n = \frac{1}{n(2n-1)\pi},\ n = 1,2,\dots .$$

The sequence $Q(h'_n,h''_n)$ has not a limit, when $n\to\infty$.

The sum of the right-hand side of relation (1) is the "best local approximation" of f, in the neighbourhood of x_0, by the linear subspace H generated by the basis $\{h^k, k=0,1,\dots,n\}$ [3]. The coefficients A_k, given by (7), are completely determined by f, x_0 and H, that is why we call number $k!\,A_k$ the kth-order H-derivative of f at x_0 and write this as $H^k f(x_0) = k!\,A_k$ [4].

Remark 2. Further generalizations of the kth-order derivative of a function at a point x_0 may be obtained by using the same idea, but starting from another asymptotic scale (for example, the scale $\{(\Phi(x_0+h)-\Phi(x_0))^k,\ k=0,1,\dots,n\}$, where Φ is a strictly monotonic function in a neighbourhood of x_0, which is continuous at x_0) ([5], [6]).

Remark 3. Similar investigations would be interesting to be done for the functions of p variables as well as the research of the connection between the coefficients of its asymptotic expansions and the various generalizations of partial derivatives.

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