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A REVERSIBLE RANDOM SEQUENCE ARISING IN THE METRIC THEORY OF THE CONTINUED FRACTION EXPANSION

MARIUS IOSIFESCU

1. INTRODUCTION

Let Ω denote the collection of irrational numbers in the unit interval $I = [0, 1]$. Consider the so-called continued fraction transformation τ of Ω defined as $\tau(\omega) = 1/\omega \pmod 1$ = fractionary part of $1/\omega$, $\omega \in \Omega$. Define N_+-valued functions a_n on Ω by $a_{n+1}(\omega) = a_1(\tau^n(\omega))$, $n \in \mathrm{N}_+ = \{1, 2, \ldots\}$, where $a_1(\omega)$ = integer part of $1/\omega$, $\omega \in \Omega$. Here τ^n denotes the nth iterate of τ. For any $n \in \mathrm{N}_+$, writing

$$[x_1] = 1/x_1,\ [x_1, \ldots, x_n] = 1/(x_1 + [x_2, \ldots, x_n]),\ n \geq 2,$$

for arbitrary indeterminates $x_i, 1 \leq i \leq n$, we have

$$\omega = \lim_{n\to\infty}[a_1(\omega), \ldots, a_n(\omega)], \omega \in \Omega,$$

and this explains the name of τ. Clearly, when I is endowed with the σ-algebra $\mathcal{B}_I$ of its Borel subsets, the $a_n, n \in \mathrm{N}_+$, are random variables defined almost everywhere with respect to any probability measure on $\mathcal{B}_I$ assigning probability 0 to the set $I - \Omega$ of rational numbers in I (thus, in particular, with respect to Lebesgue measure λ).

A great deal of work was done on the random sequence $(a_n)_{n\in\mathrm{N}_+}$ and related sequences. This is known as the metric theory of the continued fraction expansion (see, e.g., [3, Section 5.2]). The probability structure of the sequence $(a_n)_{n\in\mathrm{N}_+}$ under λ is described by the equations

$$\lambda(a_1 = i) = \frac{1}{i(i+1)},$$

$$\lambda(a_{n+1} = i | a_1, \ldots, a_n) = p_i(s_n), n \in \mathrm{N}_+, \tag{1}$$

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where

$$p_i(x) = \frac{x+1}{(x+i)(x+i+1)}, i \in \mathbf{N}_+, x \in I,$$

and $s_n = [a_n, \ldots, a_1]$. Thus, under λ, the sequence $(a_n)_{n \in \mathbf{N}_+}$ is neither independent nor Markovian. There is a probability measure γ on $\mathcal{B}_I$ which makes $(a_n)_{n \in \mathbf{N}_+}$ into a strictly stationary sequence. Known as Gauss' measure, γ is defined as

$$\gamma(A) = \frac{1}{\log 2} \int_A \frac{dx}{x+1}, \quad A \in \mathcal{B}_I.$$

It is easy to check that γ is τ-invariant, that is, $\gamma(\tau^{-1}(A)) = \gamma(A)$ for all $A \in \mathcal{B}_I$. Hence, by its very definition, $(a_n)_{n \in \mathbf{N}_+}$ is a strictly stationary sequence under γ.

2. A REVERSIBILITY PROPERTY

The aim of this paper is to prove the following result.

THEOREM. *The random sequence $(a_n)_{n \in \mathbf{N}_+}$ on $(I, \mathcal{B}_I, \gamma)$ is reversible, that is, the distributions of $(a_l : m \le l \le n)$ and $(a_{m+n-l} : m \le l \le n)$ are identical for all $m, n \in \mathbf{N}_+, m \le n$.*

The proof shall follow from a chain-of-infinite-order representation of the incomplete quotients $a_n, n \in \mathbf{N}_+$, which we are going to describe (cf. [2]). It should be noted that a direct proof (via direct computations) is possible (see [1]).

Consider the so-called natural extension τ_e of τ, which is defined as

$$\tau_e(\omega, \theta) = \left(\tau(\omega), \frac{1}{a_1(\omega) + \theta}\right), (\omega, \theta) \in \Omega^2.$$

This is a one-to-one transformation of Ω^2 with inverse

$$\tau_e^{-1}(\omega, \theta) = \left(\frac{1}{a_1(\theta) + \omega}, \tau(\theta)\right), (\omega, \theta) \in \Omega^2.$$

The transformation τ_e preserves the measure μ on $\mathcal{B}_I^2$ defined as

$$\mu(B) = \frac{1}{\log 2} \iint_B \frac{dx dy}{(1+xy)^2}, \quad B \in \mathcal{B}_I^2,$$

that is, $\mu(\tau_e^{-1}(B)) = \mu(B)$ for all $B \in \mathcal{B}_I^2$. We have

$$\mu(A \times I) = \mu(I \times A) = \gamma(A), A \in \mathcal{B}_I.$$

Define random variables $\bar{a}_l$, $l \in \mathbf{Z} = \{\ldots, -1, 0, 1, \ldots\}$, on Ω^2 by

$$\bar{a}_{l+1}(\omega, \theta) = \bar{a}_1(\tau_e^l(\omega, \theta)),$$

with τ_e^0 = identity map and

$$\bar{a}_1(\omega, \theta) = a_1(\omega), (\omega, \theta) \in \Omega^2.$$

We actually have

$$\bar{a}_n(\omega, \theta) = a_n(\omega), \bar{a}_0(\omega, \theta) = a_1(\theta), \bar{a}_{-n}(\omega, \theta) = a_{n+1}(\theta)$$

for all $n \in \mathbf{N}_+$ and $(\omega, \theta) \in \Omega^2$. By its very definition, the double infinite sequence $(\bar{a}_l)_{l \in \mathbf{Z}}$ on $(I^2, \mathcal{B}_I^2, \mu)$ is a strictly stationary one. Clearly, it is a double infinite version of $(a_n)_{n \in \mathbf{N}_+}$ under γ. It has been proved in [2] that for any $i \in \mathbf{N}_+$ and $l \in \mathbf{Z}$ we have

(2) $$\mu(\bar{a}_{l+1} = i | \bar{a}_l, \bar{a}_{l-1}, \ldots) = p_i(a) \ \mu\text{-a.s.},$$

where $a \in \Omega$ is the continued fraction with incomplete quotients $\bar{a}_l, \bar{a}_{l-1}, \ldots$. It is interesting to compare (1) and (2). The second equation emphasizes a chain-of-infinite-order structure of the incomplete quotients $a_n, n \in \mathbf{N}_+$, when properly defined on a richer probability space. For further comments see [2].

Now, coming to the proof of our theorem, we note that, by strict stationarity under μ, for fixed $m \le n$, $m, n \in \mathbf{N}_+$, the distribution of $(\bar{a}_l : m \le l \le n)$ is identical with the distribution of $(\bar{a}_{l-m-n+1} : m \le l \le n)$ (both under μ). But, by the very definition of $(\bar{a}_l)_{l \in \mathbf{Z}}$, the first distribution is identical with that of $(a_l : m \le l \le n)$, while the second one is identical with that of $(a_{m+n-l} : m \le l \le n)$ (both under γ). The proof is complete.

COROLLARY (cf. [2]). *The double infinite sequence $(\bar{a}_l)_{l \in \mathbf{Z}}$ on $(I^2, \mathcal{B}_I^2, \mu)$ is reversible, that is, the distributions of $(\bar{a}_l)_{l \in \mathbf{Z}}$ and $(\bar{a}_{-l})_{l \in \mathbf{Z}}$ are identical.*

This follows from the very definition of the $\bar{a}_l, l \in \mathbf{Z}$.

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Romanian Academy
Centre for Mathematical Statistics
Casa Academiei Române
13, Calea 13 Septembrie
RO-76100 Bucharest 5
Romania
E-mail: miosifes@valhalla.racai.ro