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A POPOVICIU-TYPE MEAN VALUE THEOREM

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PRELIMINARIES

Let $n \geq 1$ be an integer and let $[a, b] \subset \mathbf{R}$ be an interval. We shall use the following notation

$$T = \{(x_0, \ldots, x_n) \in \mathbf{R}^{n+1} \mid a \leq x_0 < \ldots < x_n \leq b\}. \tag{1}$$

Let $F : T \to \mathbf{R}$ be a function satisfying the following condition

$$\begin{cases} \bullet \min\{F(x_0,\ldots,x_n), F(x_1,\ldots,x_{n+1})\} \leq \\ \leq F(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}) \leq \\ \leq \max\{F(x_0,\ldots,x_n), F(x_1,\ldots,x_{n+1})\}, \\ \text{for all } a \leq x_0 < \ldots < x_{n+1} \leq b,\ i = 1,\ldots,n; \\ \bullet \text{if, for fixed } x_0,\ldots,x_{n+1} \text{ and for a fixed } i, \text{ at least} \\ \text{one of the two previous inequalities is an equality, then} \\ F(x_0,\ldots,x_n) = F(x_1,\ldots,x_{n+1}). \end{cases} \tag{2}$$

Let $m \in \mathbf{N}$. For a given point $(a_0,\ldots,a_n) \in T$, we consider a finite subset $M \subset T$ of the form

$$M = \{(x_j,\ldots,x_{j+n}) \in T \mid j = 0,\ldots,m; x_0 = a_0,\ x_{n+m} = a_n; \{a_0,\ldots,a_n\} \subseteq \{x_0,\ldots,x_{n+m}\}\}. \tag{3}$$

Using (2), one can show that

$$\min F(M) \leq F(a_0,\ldots,a_n) \leq \max F(M). \tag{4}$$

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MAIN RESULTS

THEOREM 1. *If* $F:T\to\mathbf{R}$ *is continuous and satisfies condition* (2), *then, for any point* $(a_0,\dots,a_n)\in T$ *and for any* $\varepsilon>0$, *there exist a point* ξ, $a_0\le\xi<a_n$, *and* $h>0, h\le\varepsilon$, *such that*

(5) $$F(a_0,\dots,a_n)=F(\xi,\xi+h,\dots,\xi+nh).$$

Proof. For the sake of simplification, we shall consider the interval [0, 1] instead of the generally used [*a*, *b*].

Case 1. We suppose that $\inf F(T)<F(a_0,\dots,a_n)<\sup F(T)$. It follows that there exist points $(\alpha_0,\dots,\alpha_n),(\beta_0,\dots,\beta_n)\in T\cap\mathbf{Q}^{n+1}$ such that

(6) $$F(\alpha_0,\dots,\alpha_n)\le F(a_0,\dots,a_n)\le F(\beta_0,\dots,\beta_n).$$

Let p_k and q_k be the denominators of α_k and, respectively, β_k, $k=0,\dots,n$. We shall use the notations:

$$h=\frac{1}{p_0p_1\cdots p_nq_0q_1\cdots q_n},\ x_i=ih,\ y_j=jh;$$

$$i=\frac{\alpha_0}{h},\frac{\alpha_0}{h}+1,\dots,\frac{\alpha_n}{h};\ j=\frac{\beta_0}{h},\frac{\beta_0}{h}+1,\dots,\frac{\beta_n}{h}.$$

By taking the denominators sufficiently large, we have $h\le\varepsilon$. From (4) we obtain

$$F(a_0,\dots,a_n)\ge\min\left\{F(ih,ih+h,\dots,ih+nh)\,\middle|\,i=\frac{\alpha_0}{h},\frac{\alpha_0}{h}+1,\dots,\frac{\alpha_n}{h}-n\right\},$$

$$F(a_0,\dots,a_n)\le\max\left\{F(jh,jh+h,\dots,jh+nh)\,\middle|\,j=\frac{\beta_0}{h},\frac{\beta_0}{h}+1,\dots,\frac{\beta_n}{h}-n\right\}.$$

Let *i*, *j*, for definiteness, $i\le j$, such that

$$F(ih,ih+h,\dots,ih+nh)\le F(a_0,\dots,a_n)\le F(jh,jh+h,\dots,jh+nh).$$

The function $g:[i,j]\to\mathbf{R}$,

$$g(t)=F(th,th+h,\dots,th+nh),$$

is continuous and

$$g(i)\le F(a_0,\dots,a_n)\le g(j);$$

it follows that there exists $c\in[i,j]$ such that $g(c)=F(a_0,\dots,a_n)$. With $\xi=ch$ we obtain

$$F(a_0,\dots,a_n)=F(\xi,\xi+h,\dots,\xi+nh).$$

Case 2. We suppose that $F(a_0,\dots,a_n)=\sup F(T)$ (or $\inf F(T)$).

By using (2), one can prove that there exists a point $(b_0,\dots,b_n)\in T\cap\mathbf{Q}^{n+1}$ such that

(7) $$F(a_0,\dots,a_n)=F(b_0,\dots,b_n).$$

We proceed as in Case 1, by using (7) instead of (6).

THEOREM 2. *If* $F:T\to\mathbf{R}$ *is continuous and satisfies condition* (2), *then, for any* $p\in\mathbf{N}^*$ *and for any* $h>0, a+nph\le b$, *there exists a point* ξ, $a\le\xi<b$, *such that*

(8) $$F(a,a+ph,\dots,a+nph)=F(\xi,\xi+h,\dots,\xi+nh).$$

Proof. We use the method of the previous proof and the inequalities

$$F(a,a+ph,\dots,a+nph)\ge\min_{j=0,\dots,np-n}F(a+jh,\dots,a+(j+n)h),$$
$$F(a,a+ph,\dots,a+nph)\le\max_{j=0,\dots,np-n}F(a+jh,\dots,a+(j+n)h).$$

□

We present below some applications.

Let $f:[a,b]\to\mathbf{R}$ be a continuous function. We use the notation

$$\Delta_h^n f(c)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(c+ih).$$

It is well known that

(9) $$[c,c+h,\dots,c+nh;f]=\frac{1}{n!\,h^n}\Delta_h^n f(c).$$

The theorem below gathers some results contained in [6].

THEOREM 3. *If the function f is continuous on an interval* [*a*, *b*] *containing the points* $a_0<\dots<a_n$, *then there exist points* $c,c+h,\dots,c+nh\in[a,b]$ *so that*

$$[a_0,\dots,a_n;f]=\frac{1}{n!\,h^n}\Delta_h^n f(c).$$

Proof. We consider the function $F:T\to\mathbf{R}$,

$$F[x_0,\dots,x_n]=[x_0,\dots,x_n;f].$$

It satisfies (2) (see [6, p. 352]). Therefore, using (9), Theorem 3 is a consequence of Theorem 1.

Theorem 3 also gives an answer to a question asked by Heinz H. Gonska and Xin-long Zhou in [1, p. 9]. Another answer is given in [3].

Some consequences of Theorem 2 are the following:

THEOREM 4. *If* $f:[a,b]\to\mathbb{R}$ *is continuous, then, for any* $p,n\in\mathbb{N}^*$ *and for any* $h>0$, $(a+nph\le b)$, *there exists a point* ξ, $a\le\xi<b$, *such that*

$$\Delta^n_{ph}f(a)=p^n\Delta^n_h f(\xi). \tag{10}$$

THEOREM 5 (P. Lévy) [8, Ex. 5.5.78, p. 231]. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuous function such that* $f(a)=f(b)$. *Then, for any positive integer p, there exists* $\xi\in[a,b]$ *such that* $f(\xi)=f(\xi+(b-a)/p)$.

This is a consequence of Theorem 4, for $h=\dfrac{b-a}{p}$, $n=1$, $\Delta_{b-a}f(a)=0$.

Some results related to Theorem 5 are:

THEOREM 6 (N. Ciorănescu) [8, Ex. 5.5.40, p. 216]. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuous function so that* $f(a)=f(b)\neq f(x)$, *for all* $x\in(a,b)$. *Then, for any* $0<d<b-a$, *there exists* $\xi\in(a,b)$ *such that* $f(\xi)=f(\xi+d)$.

THEOREM 7 [8, Ex. 5.5.54, p. 220]. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function so that the limits* $f(-\infty),f(\infty)$ *exist and* $f(-\infty)=f(\infty)$. *Then, for any* $d>0$, *there exists* $\xi\in\mathbb{R}$ *such that* $f(\xi)=f(\xi+d)$.

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