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CONTINUOUS DEPENDENCE AND FIXED POINTS FOR SOME MULTIVALUED OPERATORS

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1. INTRODUCTION

In this paper we extend the notions of I^0-continuity and uniform I^0-continuity (given by I. Del Prete and C. Esposito in [2]) for multivalued operators on metric spaces, as follows:

Let (X,d) be a metric space and denote

$$\begin{aligned} \mathcal{P}(X) &= \{A \mid A \subseteq X\} \\ P_{cl}(X) &= \{A \in \mathcal{P}(X) \mid A \neq \varnothing,\ A \text{ is closed}\} \\ D(x,A) &= \inf\{d(x,a) \mid a \in A\},\ \text{where } x \in X \text{ and } A \in \mathcal{P}(X) \end{aligned}$$

$H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{+\infty\}$ the Hausdorff-Pompeiu generalized semimetric on $\mathcal{P}(X)$, given by

$$H(A,B) := \begin{cases} \max\left\{ \sup\limits_{a\in A} D(a,B), \sup\limits_{b\in B} D(b,A) \right\}, & \text{if } A \neq \varnothing \neq B \\ 0, & \text{if } A = \varnothing = B \\ +\infty, & \text{if } A = \varnothing \neq B \text{ or } A \neq \varnothing = B. \end{cases}$$

Consider $F : X \to \mathcal{P}(X)$ a multivalued operator. Let Fix $F := \{x \in X \mid x \in F(x)\}$.

Definition 1.1. *We say that F is I^0-continuous if and only if, for every $\varepsilon > 0$, there exists $\delta > 0$ with the property that the following implication holds: $\forall x \in X$ with $D(x, F(x)) < \delta \Rightarrow \exists x^* \in FixF$ such that $d(x^*, x) < \varepsilon$.*

Now, let (Y,τ) be the topological space of the parameters and $F : X \times Y \to \mathcal{P}(X)$.

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DEFINITION 1.2. *F is called locally uniform I^0-continuous at $y_0 \in Y$ if and only if for each $\varepsilon > 0$ there exist a neighborhood V of y_0 and $\delta > 0$ with the property that for every $y \in V$ and $x \in X$ the following implication holds: $D(x, F(x,y)) < \delta \Rightarrow \exists x^* \in \mathrm{Fix}F(\cdot, y)$ such that $d(x^*, x) < \varepsilon$.*

The aim of this paper is to give some results on the continuity of the multivalued operator $p : Y \to \mathcal{P}(X)$, $p(y) = \mathrm{Fix}F(\cdot, y)$ (where $\mathrm{Fix}F(\cdot, y) = \{x \in X | x \in F(x,y)\})$.

2. MAIN RESULTS

Throughout this section d will be a metric on the nonempty set X and (Y, τ) the topological space of the parameters.

We consider the multivalued operator $F : X \times Y \to \mathcal{P}(X)$. The next result is an extension for multivalued operators of the main theorem from [3].

THEOREM 2.1. *Let $y_0 \in Y$ and assume that:*

i) *$F(x, \cdot) : Y \to \mathcal{P}(X)$ is lower semicontinuous at y_0, for every $x \in X$.*
ii) *F is locally uniform I^0-continuous at y_0.*

Then $p : Y \to \mathcal{P}(X)$ is lower semicontinuous at y_0.

Proof. We have to show that, for each $x \in p(y_0)$ and for every neighborhood $U(x)$ of x there exists a neighborhood V of y_0 such that $p(y) \cap U(x) \neq \varnothing$, $\forall y \in V$.

Let $x_0 \in p(y_0)$ be arbitrary and a neighborhood $U(x_0)$ of x_0 in X. Let $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset U(x_0)$. From *ii)* we obtain a neighborhood V_1 of y_0 and $\delta > 0$ such that the following implication holds: $\forall (x, y) \in X \times V_1$ with $D(x, F(x,y)) < \delta \Rightarrow \exists x^* \in p(y)$ such that $d(x, x^*) < \varepsilon$.

Using *i)*, we can obtain a neighborhood V_2 of y_0 such that $F(x_0) \cap B(x_0, \delta) \neq \varnothing$, for every $y \in V_2$. Let $V = V_1 \cap V_2$. Then, for each $y \in V$, we have $D(x_0, F(x_0, y)) < \delta$, so there exists $x^* \in p(y)$, with $d(x_0, x^*) < \varepsilon$. It follows that $x^* \in p(y) \cap B(x_0, \varepsilon) \subset p(y) \cap U(x_0)$. The proof is complete. □

In the next result, we shall weaken the assumption *ii)* but then we must replace *i)* with another condition.

THEOREM 2.2. *Let $y_0 \in Y$ and assume that:*

iii) *$F(x, \cdot) : Y \to \mathcal{P}(X)$ is upper semicontinuous at y_0, uniform with respect to $x \in X$.*
iv) *$F(\cdot, y_0) : X \to \mathcal{P}(X)$ is I^0-continuous.*
v) *$p(y_0)$ is compact.*

Then $p : Y \to \mathcal{P}(X)$ is upper semicontinuous at y_0.

Proof. We shall prove that for each open set $U \subseteq X$ with $U \supset p(y_0)$, there exists a neighborhood V of y_0 such that $p(y) \subset U$, for every $y \in V$. Let $U \subseteq X$ be an open set such that $U \supset p(y_0)$. From the compactness of $p(y_0)$, we deduce the existence of $\varepsilon > 0$ with the property that $\bigcup_{x \in p(y_0)} B(x,\varepsilon) \subset U$. Now, we shall demonstrate the existence of a neighborhood V of y_0 such that $p(y) \subset \bigcup_{x \in p(y_0)} B(x,\varepsilon)$, for every $y \in V$.

Clearly, condition iv) implies the existence of $\delta > 0$ such that if $x \in X$ and $D(x, F(x,y_0)) < \delta$, then there exists $x^* \in p(y_0)$ with $d(x,x^*) < \varepsilon$. Using iii), we can construct a neighborhood V of y_0 with the property that $F(x,y) \subset \bigcup_{x' \in F(x,y_0)} B(x',\delta)$, for every $(x,y) \in X \times V$. Consequently, for each $(y,x) \in V \times p(y)$ we have that $D(x, F(x,y_0)) < \delta$. Taking into account the property of δ, for each $(y,x) \in V \times p(y)$ there exists $x^* \in p(y_0)$ such that $x \in B(x^*,\varepsilon)$ or, equivalently, $p(y) \subset \bigcup_{x^* \in p(y_0)} B(x^*,\varepsilon)$, for every $y \in V$. The proof is complete. □

Remark 2.3. In a previous paper, these abstract results are applied for the classical dependence on parameters of the fixed points set for some contractive multivalued operators. For example, using Theorem 2.1, and Corollary 1 from [1], we have

Theorem 2.4. *Let $F : X \times X \to \mathcal{P}(X)$ be a multivalued operator such that:*

a) *$F(x,\cdot)$ is continuous at $y_0 \in Y$ for each $x \in X$.*

b) *$F(\cdot,y)$ is a k-contraction for every $y \in Y$.*

Then p is lower semicontinuous at y_0.

Sketch of the proof. Condition b) implies that F is uniform I^0-continuous. Since the continuity of $F(x,\cdot) : Y \to \mathcal{P}(X)$ (where the topology on $\mathcal{P}(X)$ is given by the generalized semimetric H) implies i) from Theorem 2.1, the conclusion follows.

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