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REMINDER ON THE HISTORY OF SPLINE FUNCTIONS

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At the end of the twentieth century, which sometimes is called the century of information processing, numerical analysis has been increasingly employed in the areas of applied mathematics. Among all fields, nowadays the spline theory is probably the most active field in the approximation theory and plays an important part in present-day mathematics and its technological applications.

Since spline functions are easy to evaluate and manipulate on computer, being also able to be used with the remarkable theoretical developments, a lot of important problems have been investigated and solved by using spline functions. These include, for example, data fitting, functions approximation, numerical integration (quadrature) and differentiation, numerical solutions of operator equations, optimal control problems, computation of the eigenvalues and eigenfunctions of operators, numerical methods of probabilities and statistics, computer-aided geometric design, computerised tomography, wavelets theory, etc.

Today it seems to be a difficult adventure to propose a list of literature on spline functions and their applications, because in the last decades over 350 books, monographs and conference reports have been published.

There are also thousands of original papers and more than 400 dissertations for doctor's degree on various aspects of the spline functions.

To underline the efficiency and also the fascination of splines in modern applied mathematics, let us remember the following words of the American mathematician Philip J. Davis: "Spline approximations contain the delicious paradox of Prokofiev's Classical Symphony: it seems as though it might have been written several centuries ago, but of course it could not have been" (Symposium on Approximation of Functions, Warren, Michigan, USA, 1964).

For a detailed presentation of the problem on spline functions we refer to the monographs [2], [3], [15] and [26], and for an exhaustive literature we refer to [17].

Generally, I. J. Schoenberg (1903–1990) is regarded as the father of splines, particularly on account of his pioneering paper [23]. If there is a father of splines, there also has to be a grandfather or a great...grandfather.

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Even Schoenberg himself states that "B-splines were probably known to Hermite and certainly to Peano" (see [24], p. 1) and further "B-splines were already known to Laplace (1820)" (see [24], p. 11). Still further back in history, Schoenberg writes [24] that certain splines have their seed in "exponential Euler Splines" based on the generating functions studied by Euler (1755).

The purpose of this paper is to point out briefly that the spline functions have been considered in a variety of publications, discovered and rediscovered independently by many mathematicians, particularly in the period 1895–1945, but so far they seem to have been mainly overlooked. As we shall further underline, an essential contribution in the development of spline function theory was brought by the Romanian mathematicians T. Popoviciu (1906–1975) and D. V. Ionescu (1901–1984). Their contributions preceded the paper [24] of I. J. Schoenberg, himself born in Romania (Galaţi, 1903) and graduated in 1926 at the University of Iaşi. In his fascinating book – *Mathematical Time Exposured*, AMS Inc., 1982 (translated in Romanian in 1989) – and also in paper [25], Schoenberg underlined the contributions of the Romanian mathematicians T. Popoviciu and D. V. Ionescu in the field of spline approximation.

Following the excellent survey paper of Butzer, Schmidt and Stark [5], we shall briefly mention in the development of spline function theory the contribution of L. Maurer (1896), M. Learch (1908), A. Sommerfeld (1904 and 1928), C. Runge (1904), H. L. Rietz (1924), T. Popoviciu (1935), J. E. Fjeldstaad (1937), K. Fränz (1940) and D. V. Ionescu (1952), several other authors involved in splines being only summarily presented.

Nominating I. J. Schoenberg as the father of splines, it will be seen that K. Fränz could be called the "engineering grandfather of splines" and A. Sommerfeld deserves the attribute of "great-grandfather of splines", being the first to give a geometric interpretation of B-splines as well as to draw some spline curves.

The central B-spline of order $n \in \mathbb{N}$, *alias* basic spline curves, *alias* spline frequency function, *alias* fundamental spline function, is defined by

$$M_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right]^n e^{iux}\,du = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\frac{u}{2}}{\frac{u}{2}}\right]^n \cos(ux)\,dx, \tag{1}$$

where the Fourier transform on $\mathbb{R}$ can be replaced by the Fourier cosine transform on $(0,\infty)$, since the sin c-function defined by

$$\sin c(x) = \frac{\sin x}{x} \text{ for } x \neq 0 \text{ and } = 1 \text{ for } x = 0$$

is odd.

It is shown that

$$M_n(x) = \int_{x-1}^{x+1} M_{n-1}(u)\,du, \tag{2}$$

where M_n has support $\left[-\frac{n}{2},\frac{n}{2}\right]$ with $M_n(x) > 0, M_1(x) = 1$ for $x \in \left[-\frac{1}{2},\frac{1}{2}\right]$ and $= 0$ for $x \in \left[-\frac{1}{2},\frac{1}{2}\right]$ and further

$$\int_{-\infty}^{+\infty} M_n(x)\,dx = 1,\ n \in \mathbb{N}. \tag{3}$$

Whereas expression (1) is useful for mathematical purposes, as regards the applications it is more practical to consider the explicit representation

$$M_n(x) = \frac{1}{(n-1)!}\sum_{k=0}^{\infty}(-1)^k C_n^k\left(x+\frac{n}{2}-k\right)_+^{n-1}. \tag{4}$$

The functions M_n form a basis in the sense that every spline function S_n of order n (namely a function of class $C^{n-2}(R)$ such that on each interval $\left(k-\frac{n}{2},k+1-\frac{n}{2}\right)$, $k \in Z$, reduces to a polynomial of degree $n-1$) can be uniquely represented in the form

$$S_n(x) = \sum_{k=-\infty}^{+\infty} c_k M_n(x-k) \tag{5}$$

with appropriate constant coefficients c_k. Conversely, any such series represents a spline of order n.

A point of interest in spline theory is the approximation of a function f from its values $f\left(\frac{k}{W}\right)$ taken on the nodes $\frac{k}{W}$ equally spaced on the real axis $\mathbb{R}$, in the form of the operators

$$A_W^n f(x) = \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{W}\right) M_n(Wx-k),\ x \in \mathbb{R} \tag{6}$$

as $W \to \infty$, $n \in \mathbb{N}$ being fixed. It is known that, provided $f \in C(R)$,

$$A_W^n f(x) \to f(x),\ W \to \infty. \tag{7}$$

With regard to the literature, Schoenberg himself [23] defines M_n via the Fourier transform (1) and observes (4). He adds that the representation of (1) in the form (4) is essentially due to Laplace, a fact taken up by J. V. Uspensky (1938) [29, pp. 277–278], and that S. Bochner (1936) [1] worked out $M_n(x)$ for $n = 1, 2, 3$.

He also cites the works of W. A. Jenkins (1926) and T. N. E. Greville (1944) [9]. Regarding the transform connection of B-splines with the sin c-function, a large number of research articles have been done in the past 50 years, but the best information is to be found in [3], [24] and [26].

L. Maurer [14] defined in 1896 the nth integral mean $f_n(x)$ of an integrable and bounded function f on R successively by

$$(8)\qquad f_n(x) = \frac{1}{2h}\int_{-h}^{+h} f_{n-1}(x+t)\,dt,\ f_0(x) = f(x),\ h \in R_+,$$

and showed that the following relation held

$$(9)\qquad f_n(x) = \frac{1}{2h}\int_{-\infty}^{+\infty} f(x+u)M_n\left(\frac{u}{2h}\right)du = \int_{-\infty}^{+\infty} f(2hu)M_n\left(\frac{x}{2h}-u\right)du.$$

Also, he proved that $f_n(x) \to f(x)$, $n \to \infty$ and $h \to 0$.

Observe that one can regard $f_n(x)$ in the form (9) as a continuous version of the discrete spline-approximation $A^n_W f$ considered in (6) with $W = \frac{1}{2h}$.

He also established that

$$(10)\qquad M_n(x) \simeq \sqrt{\frac{6}{\pi}}\, e^{-\frac{6x^2}{n}},\ n \to \infty,$$

so that the $M_n(x)$ approximate the Gauss frequency distribution.

A. Sommerfeld [27] was interested in this aspect and noted Maurer [14] for the precise mathematical details shown in 1904, namely, that the approximation (10) might be also rewritten in the form

$$(11)\qquad \sqrt{\frac{n}{6}}\,M_n\left(\sqrt{\frac{n}{6}}\,x\right) \to \frac{1}{\sqrt{\pi}}e^{-x^2},\ n \to \infty,$$

locally uniform in x, a result established in a much more general frame by Curry-Schoenberg [6, p. 104] in 1966. Sommerfeld's major contribution with regard to splines is that he drew the first four spline curves $M_n(x)$ for $n = 1, 2, 3, 4$, and compared them with $\sqrt{\frac{6}{\pi}}e^{-6x^2}$, giving a geometric interpretation of $M_n(x)$.

It is G. Pólya's merit to have given a complete and precise derivation of the Sommerfeld representation (11) in a paper from 1913 on the evaluation of definite integral – a part of his doctorate thesis presented in 1912 in Budapest.

There are a number of mathematicians concerned, at least indirectly, with B-splines.

M. Lerch [13] in 1908, using methods related to those of Maurer and defining the functions

$$H_n(x) = \frac{2}{\pi}\int_0^{\infty}\left[\frac{\sin t}{t}\right]^n \frac{\sin 2xt}{t}\,dt,$$

showed that this function could be put in the form

$$(12)\qquad H_n(x) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k C_n^k\left(x+\frac{n}{2}-k\right)^n sign\left(x+\frac{n}{2}-k\right)$$

and proved the following important formula

$$H'_n(x) = 2M_n(x).$$

H. L. Rietz [21] in 1928 established the same formula, using the deep reasons of the probabilities tackled by Sommerfeld.

C. Runge [22, pp. 192–196] in 1904 discussed the interpolation of periodic real functions by periodic spline functions (without calling them splines) with equidistant knots for the purpose of getting improved values of the Fourier coefficients of the given function whose values were known only on the knots.

W. A. Quade and L. Collatz [20] in 1938 greatly elaborated Runge's idea, for the same reason as Runge. In the process they derived and anticipated many results concerning spline functions with equidistant knots, including an analysis of the order of approximation thus obtained.

T. Popoviciu [18, pp. 96–105] in 1941 used spline functions (also without calling them splines) for the purpose for which, it was much later observed, they were so eminently suited: the approximation of functions. Popoviciu introduced spline functions of degree n with arbitrary knots, which he called elementary function of degree n. In particular, he showed that a continuous nonconcave function of order n, in a finite interval $[a, b]$, is the uniform limit of elementary functions of order n that are also nonconcave of order n in $[a, b]$ [18, Theorem 6, p. 96].

J. E. Fjeldstaad [7] in 1937 gave an explicit representation of the central B-splines that differed essentially from that of Sommerfeld's, both with regard to the proof and to the form of representation. In fact, Fjeldstaad's representation for the B-splines was

$$M_n(x) = \frac{1}{(2n-1)!}\sum_{k=0}^{n}(-1)^k C_n^k\left|x+\frac{n}{2}-k\right|^{n-1} sign\left(x+\frac{n}{2}-k\right)^n.$$

A further approach to central B-splines is due to K. Fränz [8] in 1940, in a paper on signal and noise voltage. Disregarding the technological background, Fränz defined the functions

$$f_p(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f_{p-1}(u)\,du, \quad p = 2, 3, \ldots \tag{13}$$

$f_1(x) = 1$ for $|x| \le \frac{1}{2}$, $f_1(x) = 0$ for $|x| > \frac{1}{2}$. Then he noted that f_p is $(p-2)$-continuously differentiable and consists of polynomials of degree $p-1$ in the intervals $\left(r - \frac{p}{2}, r + 1 - \frac{p}{2}\right), r = 0, 1, \ldots, p-1.$

On the final investigation, Fränz arrived at the B-splines $f_p(\ ) = M_p(x)$. Fränz's article gave rise not only to further engineering research but also to research in probabilities and computational topics.

Beginning with the year 1950, D. V. Ionescu ([10–12], [16]) has created a new general method of constructing the approximation formulas of mathematical analysis, called by him the method of function φ. This method is known today in mathematical literature as the "D. V. Ionescu constructing method of spline functions". The starting point of his method is the classical Green's formula. Similar approaches were attempted by J. Radon in 1935 and A. Ghizzetti in 1954, but the D. V. Ionescu method, by its general character, is applicable to all linear approximating formulas of analysis in one or more variables. D. V. Ionescu method mainly consists in associating to any approximating formula a boundary value problem on an ordinary or partially differential equation, the boundary conditions being suitably chosen, according to the formula to be established. The solution of this boundary problem, usually denoted by φ, is a spline function and it generates the coefficients, the knots as well as its remainder $R[f]$, expressed under the form

$$R[f] = \int_a^b \varphi(x) f^{(n)}(x)\,dx, \quad f \in C^n[a, b]. \tag{14}$$

The representation (14) was also found, but only for particular cases, by G. Peano (1913), L. Tschakaloff (1936), G. Kowalevski (1932), and R. von Mises (1936). However, the D. V. Ionescu constructing method of splines differs from these by its generality and constructiveness, the function φ being a solution of the boundary value problem, which is ingeniously solved.

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