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ON THE CONVERGENCE OF A CLASS OF NUMERICAL DIFFERENTIATION FORMULAS

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1. PRELIMINARIES

Given a strictly increasing sequence of natural numbers $(j_n)_{n\geq 1}$, let us consider a node matrix

$$\{x_n^k : n \geq 1,\ \ 1 \leq k \leq j_n\}$$

with $-1 \leq x_n^1 < x_n^2 < \ldots < x_n^{j_n} \leq 1, n \geq 1$, and a matrix of real coefficients

$$\{a_n^k : n \geq 1,\ \ 1 \leq k \leq j_n\}.$$

Denote by C_1 the linear space of all functions $f: [-1, 1] \to \mathbf{R}$ which are continuous together with their derivative of the first order. For each function f in C_1 put

$$\|f\|_1 = \|f'\| + |f(0)|,$$

where $\|\cdot\|$ means the uniform norm, and remark that the linear space C_1 endowed with the norm $\|\cdot\|_1$ becomes a Banach space.

Now, define the linear functionals $D_n : C_1 \to \mathbf{R}$, $n \geq 1$, by

$$D_n f = \sum_{k=1}^{j_n} a_n^k f(x_n^k),\ \ f \in C_1. \tag{1}$$

Let us prove that D_n is a continuous functional, too, for each $n \geq 1$. Using the classical Lagrange mean value theorem, for each node $x_n^k, 1 \leq k \leq j_n$, there exists a point t_n^k between 0 and x_n^k so that

$$f(x_n^k) = f(0) + x_n^k f'(t_n^k),$$

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which implies

$$\left|f(x_n^k)\right| \le |f(0)| + \|f'\| = \|f\|_1 .$$

Further, using (1), we get

$$|D_n f| \le \sum_{k=1}^{j_n} \left|a_n^k\right| \left|f(x_n^k)\right| \le \left(\sum_{k=1}^{j_n} \left|a_n^k\right|\right) \cdot \|f\|_1,$$

which proves the continuity of D_n.

Let us consider the following *numerical differentiation formulas*

$$f'(0) = D_n f + R_n f,\ f \in C_1,\ n \ge 1, \tag{2}$$

where $R_n f$, $n \ge 1$, are the rests of the formulas (2).

Denote, usually, by $\mathcal{P}_m$ the space of all polynomials which have the degree at most m.

In what follows, we suppose that the numerical differentiation formulas (2) are of *interpolatory type*, that is, $R_n f = 0$ for all polynomials f in $\mathcal{P}_m$, $m = j_n - 1$. In this case, putting

$$w_n(x) = \left(x - x_n^1\right)\ldots\left(x - x_n^{j_n}\right) \text{ and } l_n^k(x) = \frac{w_n(x)}{\left(x - x_n^k\right) w_n'\left(x_n^k\right)},$$

by the equalities

$$D_n\left(l_n^k\right) = \left(l_n^k\right)'(0), \quad n \ge 1, \quad 1 \le k \le j_n,$$

we obtain

$$a_n^k = \begin{cases} -\dfrac{w_n'(0) x_n^k + w_n(0)}{\left(x_n^k\right)^2 w_n'\left(x_n^k\right)}, & \text{if } x_n^k \ne 0 \\[2ex] \dfrac{w_n''(0)}{2 w_n'(0)}, & \text{if } x_n^k = 0. \end{cases} \tag{3}$$

2. EVALUATING THE RESTS OF FORMULAS (2)

Since $D_n(P) = P'(0)$ for each polynomial P in $\mathcal{P}_{j_n - 1}$, we get for all functions f in C_1

$$\left|D_n f - f'(0)\right| = \left|D_n(f - P) + P'(0) - f'(0)\right| \le \sum_{k=1}^{j_n} \left|a_n^k\right| \cdot \left|f(x_n^k) - P(x_n^k)\right| + \left|f'(0) - P'(0)\right|,$$

therefore

$$\left|D_n f - f'(0)\right| \le \left(\sum_{k=1}^{j_n} \left|a_n^k\right|\right) \cdot \|f - P\| + \|f' - P'\|, \quad \forall P \in \mathcal{P}_{j_n - 1}. \tag{4}$$

Given a continuous function $g : [-1, 1] \to \mathbf{R}$ and a natural number $m \ge 1$, denote by $E_m(g)$ *the degree of approximation* of g by algebraic polynomials of the degree at most m, that is,

$$E_m(g) = \inf\left\{\|g - P\| : P \in \mathcal{P}_m\right\}.$$

Let $\omega(g;\cdot) : [0,2] \to \mathbf{R}$ be the modulus of continuity of g, namely,

$$\omega(g; h) = \max\left\{|g(x + t) - g(x)| : |x| \le 1, |x + t| \le 1, |t| \le h\right\}, \quad 0 \le h \le 2.$$

It follows from [3] that, for all g in C_1, the inequalities

$$E_{m+1}(g) \le \frac{A}{m} \cdot \omega\left(g'; \frac{1}{m}\right) \text{ and} \tag{5}$$

$$E_m(g') \le B \cdot \omega\left(g'; \frac{1}{m}\right) \tag{6}$$

hold, where A and B are real constants, which do not depend on n and g.

Now, by (4), (5) and (6) we deduce

$$\left|D_n f - f'(0)\right| \le \left(\sum_{k=1}^{j_n} \left|a_n^k\right|\right) \cdot E_{j_n - 1}(f) + E_{j_n - 2}(f') \le$$

$$\le \left(\sum_{k=1}^{j_n} \left|a_n^k\right|\right) \frac{A}{j_n - 2} \omega\left(f'; \frac{1}{j_n - 2}\right) + B\omega\left(f'; \frac{1}{j_n - 2}\right).$$

According to the properties of the modulus of continuity, we obtain

$$\left|D_n f - f'(0)\right| \le M\left(1 + \frac{1}{j_n} \sum_{k=1}^{j_n} \left|a_n^k\right|\right) \cdot \omega\left(f'; \frac{1}{j_n}\right), \tag{7}$$

for each $n \ge n_0$, where M and n_0 do not depend on n and f.

3. THE CONVERGENCE OF NUMERICAL DIFFERENTIATION FORMULAS (2)

THEOREM 1. *If* $\frac{1}{j_n} \sum_{k=1}^{j_n} \left|a_n^k\right| \in O(1)$, *then the numerical differentiation formulas* (2) *are convergent for each* f *in* C_1, *i.e.,* $\lim_{n\to\infty} D_n f = f'(0)$.

Proof. The conclusion follows from the boundedness of the sequence $\left(\frac{1}{j_n}\sum_{k=1}^{j_n}\left|a_n^k\right|\right)_{n\ge 1}$, using the inequality (7) and the equality $\lim_{n\to\infty}\omega\left(f';\frac{1}{j_n}\right)=0$.

4. A SPECIAL CASE INVOLVING JACOBI NODES

Let $P_n^{(\alpha)}$, $n \ge 1$, $\alpha > -1$, be Jacobi ultraspheric polynomials and $w_n(x) = x(1-x^2)P_{2n-2}^{(\alpha)}(x)$, $j_n = 2n+1$. In this case, the nodes of the nth row of the node matrix are

$$-1 = x_n^1 < x_n^2 < \ldots < x_n^n < x_n^{n+1} = 0 < x_n^{n+2} < \ldots < x_n^{2n} < x_n^{2n+1} = 1.$$

We put x_k for x_n^{n+1+k}, $0 \le k \le n$, and deduce that $x_n^{n+1-k} = -x_n^{n+1+k} = -x_k$, $1 \le k \le n$, $x_0 = 0, x_n = 1$. Similarly, we put a_k for a_n^{n+1+k}, $0 \le k \le n$; using (3), we obtain $a_0 = 0$ and $a_n^{n+1+k} = -a_k$, $1 \le k \le n$.

By [2] we get

(8)
$$\begin{cases} |a_k| \sim \frac{1}{n}\cdot\frac{1}{x_k^2}\cdot\left(\frac{n-k}{n}\right)^{\alpha-\frac{1}{2}}, & 1 \le k \le n-1; \\ |a_n| \sim n^{-\alpha-\frac{1}{2}} \text{ and} & \\ \sum_{k=1}^{n}\frac{1}{x_k^2} \sim n^2, & \end{cases}$$

where $a_n \sim b_n$, $b_n \ne 0$, means that there exist two real constants α, β which do not depend on n so that $0 < \alpha \le |a_n / b_n| \le \beta$ for all $n \ge 1$.

In what follows M_s, $s \ge 1$, are real numbers which do not depend on n.

So we have

(9)
$$\frac{1}{j_n}\sum_{k=1}^{j_n}\left|a_n^k\right| \le \frac{M_1}{n^2}\left[n^{\frac{1}{2}-\alpha} + \sum_{k=1}^{n-1}\frac{1}{x_k^2}\left(\frac{n-k}{n}\right)^{\alpha-\frac{1}{2}}\right].$$

If $\alpha \ge \frac{1}{2}$, it results $\left(\frac{n-k}{n}\right)^{\alpha-\frac{1}{2}} \le 1$, $n^{\frac{1}{2}-\alpha} \le 1$ and, using (8) and (9), we get

(10)
$$\frac{1}{j_n}\sum_{k=1}^{j_n}\left|a_n^k\right| = O(1).$$

If $-\frac{1}{2} \le \alpha < \frac{1}{2}$, we have $\left(\frac{n-k}{n}\right)^{\alpha-\frac{1}{2}} \le \frac{n}{n-k}$, $n^{\frac{1}{2}-\alpha} \le n$ and $x_k \sim \frac{k}{n}$ (see [2], [3]) so that from (9) we get

$$\frac{1}{j_n}\sum_{k=1}^{j_n}\left|a_n^k\right| \le M_1\left[\frac{1}{n} + \sum_{k=1}^{n-1}\frac{n}{k^2(n-k)}\right] \le M_2\sum_{k=1}^{n-1}\frac{n-k+k}{k^2(n-k)} =$$

$$= M_2\left[\sum_{k=1}^{n-1}\frac{1}{k^2} + \sum_{k=1}^{n-1}\frac{1}{k(n-k)}\right] \le M_2\left[M_3 + \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{k}+\frac{1}{n-k}\right)\right] =$$

$$= M_4 + \frac{2}{n}M_2\sum_{k=1}^{n-1}\frac{1}{k} \le M_4 + M_5\frac{\ln n}{n} \le M_6,$$

which leads to relation (10), too.

THEOREM 2. *If* $\alpha \ge -\frac{1}{2}$, *then the numerical differentiation formulas* (2) *are convergent for all* f *in* C_1.

Proof. Use Theorem 1 and relation (10).

Remarks. (i) R. A. Lorenz [1] showed, using another proof, that $D_n f \to f'(0)$ for each f in C_1, if $\alpha = \frac{1}{2}$.

(ii) Also using other arguments, A. I. Mitrea proved in [2] that $D_n f \to f'(0)$ for all f in C_1 if $\alpha \ge \frac{1}{2}$ and $D_n f \to f'(0)$ if $-\frac{1}{2} \le \alpha < \frac{1}{2}$ for each f in C_1, whose derivative f' satisfies the Dini-Lipschitz condition $\lim_{\delta\to 0}\omega(f';\delta)\cdot\ln\delta = 0$.

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