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A METHOD FOR OBTAINING ITERATIVE FORMULAS OF HIGHER ORDER FOR ROOTS OF EQUATIONS

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1. INTRODUCTION

Formulas of the class which use information at only one point are naturally called one-point formulas. We shall consider only stationary one-point formulas which have the form

$$x_{n+1} = F(x_n), \tag{1}$$

with $\alpha = F(\alpha)$, if the method converges, where α is the root of the real or complex equation $f(x) = 0$.

For the iterative method (1) which converges to α, we say it is of order k if

$$|x_{n+1} - \alpha| = 0\left(|x_n - \alpha|^k\right), \ n \to \infty. \tag{2}$$

If the function $F(x)$ is k-times differentiable in a neighborhood of the limit point $x = \alpha$, then [3] the iterative method (1) is of order k if and only if

$$F(\alpha) = \alpha, F'(\alpha) = F''(\alpha) = \ldots = F^{(k-1)}(\alpha) = 0, F^{(k)}(\alpha) \neq 0. \tag{3}$$

In Section 2 we give some results which represent the answers of the following question: If we have a method of order k, how can we obtain from it a method of order $k+1$?

In Section 3, a family of iterative functions for finding root α is derived. The family includes the functions presented in Section 2.

2. HIGHER ORDER METHODS

THEOREM 1 [4]. *Let* (1) *be an iterative method of order* $k(\geq 2)$ *and let the function* $F(x)$ *be* $k+1$*-times differentiable in a neighborhood of the limit point* $x = \alpha$. *Then*

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(4) $$x_{n+1} = F(x_n) - \frac{1}{k} F'(x_n)(x_n - F(x_n)) =$$
$$= x_n - (x_n - F(x_n))\left(1 + \frac{1}{k} F'(x_n)\right), \; n = 0, 1, 2, \dots$$

is an iterative method of order at least $k + 1$. □

THEOREM 2 [1]. *Let* (1) *be an iterative method of order* k. *Let the function* $F(x)$ *be* $k+1$*-times differentiable in a neighborhood of the limit point* $x = \alpha$ *and let* $F'(\alpha) \neq 0$. *Then*

(5) $$x_{n+1} = \frac{F(x_n) - \frac{1}{k} F'(x_n) x_n}{1 - \frac{1}{k} F'(x_n)} = x_n - \frac{x_n - F(x_n)}{1 - \frac{1}{k} F'(x_n)}, \; n = 0, 1, 2, \dots$$

is an iterative method of order at least $k + 1$. □

THEOREM 3 [4]. *Let* (1) *be an iterative method of order* k. *Let the function* $F(x)$ *be* $k+1$*-times differentiable in a neighborhood of the limit point* $x = \alpha$ *and let* $F'(\alpha) \neq 1$. *Then*

(6) $$x_{n+1} = F(x_n) - \frac{1}{k} F'(x_n)\left(\frac{x_n - F(x_n)}{1 - F'(x_n)}\right);$$

that is,

(7) $$x_{n+1} = x_n - \left(1 + \frac{1}{k}\left(\frac{F'(x_n)}{1 - F'(x_n)}\right)\right)(x_n - F(x_n)), \; n = 0, 1, 2, \dots$$

is an iterative method of order at least $k + 1$. □

Remark 1. *1. If* (1) *represents Newton's method for finding simple roots of the equation* $f(x) = 0$, *namely,*

(8) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \dots,$$

which means that

$$F(x_n) = x_n - \frac{f(x_n)}{f'(x_n)},$$

then from (4), (5) *and* (6) *we obtain the following methods, respectively:*

(9) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \cdot \frac{2(f'(x_n))^2 + f(x_n) f''(x_n)}{2(f'(x_n))^2},$$

which is known as Chebyshev's iterative method,

(10) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \cdot \frac{2(f'(x_n))^2}{2(f'(x_n))^2 - f(x_n) f''(x_n)},$$

which is known as Halley's method, and

(11) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \cdot \frac{2(f'(x_n))^2 - f(x_n) f''(x_n)}{2(f'(x_n))^2 - 2 f(x_n) f''(x_n)},$$

($n = 0, 1, 2, \dots$).

The order of these methods is at least 3 *but, since they do not involve derivatives of f higher than the second order, their order of convergence cannot exceed* 3 (see [3]).

2. *In* [2] *it is presented a family of transformations*

(12) $$T_m(x) = x - \frac{f(x)}{f'(x)} \cdot \frac{\sum_{k=0}^{m} a_k [f'(x)]^{2m-2k} [f(x)]^k [f''(x)]^k}{\sum_{k=0}^{m} b_k [f'(x)]^{2m-2k} [f(x)]^k [f''(x)]^k},$$

where $m \in \mathbf{N}^*$ *and* $a_k, b_k \in \mathbf{R}$, *which includes those of Newton and Halley, and which accelerates the convergence of the ratios of consecutive Fibonacci numbers, for some values of* a_k *and* b_k *to* φ *(the golden number).*

3. A NEW METHOD

THEOREM 4. *Let* (1) *be an iterative method of order* $k (\geq 2)$. *Let the function* $F(x)$ *be* $k+1$*-times differentiable in a neighborhood of the limit point* $x = \alpha$ *and let s be a finite parameter such that* $1 - F'(\alpha)\left(s + \frac{1}{k}\right) \neq 0$. *Then*

(13) $$x_{n+1} = x_n - (x_n - F(x_n)) \cdot \frac{1 - sF'(x_n)}{1 - \left(s + \frac{1}{k}\right) F'(x_n)}, \; n = 0, 1, 2, \dots$$

is an iterative method of order at least $k + 1$.

Proof. In the method (13) the iterative function is

$$G(x) = x - (x - F(x)) \cdot \frac{1 - sF'(x_n)}{1 - \left(s + \frac{1}{k}\right)F'(x_n)} = \tag{14}$$

$$= F(x) - \frac{1}{k} F'(x) \cdot \left(\frac{x - F(x)}{1 - \left(s + \frac{1}{k}\right)F'(x_n)} \right).$$

For the function $G(x)$ we shall prove that

$$G(\alpha) = \alpha, G'(\alpha) = G''(\alpha) = \ldots = G^{(k)}(\alpha) = 0. \tag{15}$$

By hypothesis, (1) is an iterative method of order k and, therefore, relations (3) hold.

We obtain from (14)

$$G(\alpha) = \alpha \tag{16}$$

and

$$G^{(r)}(x) = F^{(r)}(x) - \frac{1}{k}\left[F^{(r+1)}(x)\left(\frac{x - F(x)}{1 - \left(s + \frac{1}{k}\right)F'(x)} \right) + \binom{r}{1} F^{(r)}(x) \cdot \right.$$

$$\cdot \left(\frac{x - F(x)}{1 - \left(s + \frac{1}{k}\right)F'(x)} \right)' + \binom{r}{2} F^{(r-1)}(x) \left(\frac{x - F(x)}{1 - \left(s + \frac{1}{k}\right)F'(x)} \right)'' + \ldots + \tag{17}$$

$$\left. + F'(x) \left(\frac{x - F(x)}{1 - \left(s + \frac{1}{k}\right)F'(x)} \right)^{(r)} \right].$$

As regards relations (3), we obtain from (17)

$$G^{(r)}(\alpha) = 0 \text{ for } r = 1, 2, \ldots, k - 1. \tag{18}$$

Since

$$\left(\frac{x - F(x)}{1 - \left(s + \frac{1}{k}\right)F'(x)} \right)' = \frac{(1 - F'(x))\left[1 - \left(s + \frac{1}{k}\right)F'(x)\right] + (x - F(x))\left(s + \frac{1}{k}\right)F''(x)}{\left[1 - \left(s + \frac{1}{k}\right)F'(x)\right]^2},$$

we obtain

$$G^{(k)}(a) = F^{(k)}(a) - \frac{1}{k}\left[kF^{(k)}(a)\right] = 0;$$

hence conditions (15) are fulfilled. □

Remark 2. *For* $s = -\frac{1}{k}$ *we obtain the iterative method* (4), *for* $s = 0$ *we obtain the iterative method* (5) *and for* $s = 1 - \frac{1}{k}$ *we obtain the iterative method* (6).

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