REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Tome XXVI, N^{os} 1–2, 1997, pp. 137–148

APPROXIMATION BY SPLINE FUNCTIONS OF THE SOLUTION OF A BILOCAL LINEAR PROBLEM

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In the last years the theory of spline functions has become an important tool in the numerical solving of some problems for differential equations (see, for instance, [1], [2], [4]).

In this paper we shall define a space of spline functions of degree 5 which can be used to approximate the solution of a bilocal linear problem.

Let $n \geq 3$ be a natural number and let

$$\Delta_n: -\infty = t_{-1} < a = t_0 < t_1 < \ldots < t_n = b < t_{n+1} = +\infty$$

be a division of the real axis.

Denote by $S_5(\Delta_n)$ the set of all functions $s : \mathbf{R} \to \mathbf{R}$ having the following properties:

1^0 $s \in C^4(\mathbf{R})$.

2^0 $s|_{I_k} \in \mathscr{P}_5$, $I_k = [t_{k-1}, t_k)$, $k = 1, 2, \ldots, n$.

3^0 $s|_{I_0} \in \mathscr{P}_3$, $s|_{I_{n+1}} \in \mathscr{P}_3$, $I_0 = (t_{-1}, t_0)$, $I_{n+1} = [t_n, t_{n+1})$.

THEOREM 1. *If $s \in S_5(\Delta_n)$, then*

$$s(t) = \sum_{i=0}^{3} A_i t^i + \sum_{k=0}^{n} a_k (t - t_k)_+^5, \quad t \in \mathbf{R}, \tag{1}$$

where

$$\sum_{k=0}^{n} a_k = 0 \quad and \quad \sum_{k=0}^{n} a_k t_k = 0. \tag{2}$$

Proof. Let $s \in S_5(\Delta_n)$. If $t \geq b$, then $s^{(4)}(t) = 0$ so that

$$0 = 5! \sum_{k=0}^{n} a_k (t - t_k)_+ = 5! \sum_{k=0}^{n} a_k (t - t_k)_+ \text{ because}$$

AMS Subject Classification: 65D07, 65L05, 65L10.

$$(t-t_k)_+ = \begin{cases} 0 & \text{for } t < t_k \\ t-t_k & \text{for } t \ge t_k \end{cases}.$$

Consequently $\sum_{k=0}^{n} a_k = 0$ and $\sum_{k=0}^{n} a_k t_k = 0$. □

THEOREM 2. a) *If $f: \mathbf{R} \to \mathbf{R}$ verifies the conditions*

(3) $$f(a) = \alpha_1,\ f(b) = \beta_1,\ f''(t_k) = \lambda_k,\ k = 0, 1,\dots, n,$$

then there exists a unique spline function $s_f \in S_5(\Delta_n)$ such that

(4) $$s_f(a) = \alpha_1,\ s_f(b) = \beta_1,\ s_f''(t_k) = \lambda_k,\ k = 0, 1,\dots, n.$$

b) *If $h: \mathbf{R} \to \mathbf{R}$ verifies the conditions*

(5) $$h(a) = \alpha_2,\ h'(a) = \beta_2,\ h''(t_k) = \mu_k,\ k = 0, 1,\dots, n,$$

then there exists a unique spline function $s_h \in S_5(\Delta_n)$ such that

(6) $$s_h(a) = \alpha_2,\ s_h'(a) = \beta_2,\ s_h''(t_k) = \mu_k,\ k = 0, 1,\dots, n.$$

Proof. a) Using the representation (1) and taking into account conditions (4), we obtain the system

$$A_0 + A_1 a + A_2 a^2 + A_3 a^3 = \alpha_1,$$

$$A_0 + A_1 b + A_2 b^2 + A_3 b^3 + \sum_{k=0}^{n-1} a_k (b - t_k)^5 = \beta_1,$$

(7) $$2A_2 + 6A_3 t_j + 20 \sum_{k=0}^{n} a_k (t_j - t_k)_+^3 = \lambda_j;\ \ j = \overline{0,n},$$

$$\sum_{k=0}^{n} a_k = 0;\ \sum_{k=0}^{n} a_k t_k = 0$$

of $n + 5$ equations with $n + 5$ unknowns: $A_0, A_1, A_2, A_3, a_0, a_1,\dots, a_n$.

The system (7) has a unique solution if and only if the associated homogeneous system (obtained for $\alpha_1 = \beta_1 = 0, \lambda_k = 0, k = 0, 1,\dots, n$) has only the trivial solution.

Suppose that $s \in S_5(\Delta_n)$ verifies the homogeneous conditions (4) (i.e., $\alpha_1 = \beta_1 = 0, \lambda_k'' = 0, k = 0, 1,\dots, n$). Then we have

$$\int_a^b \left[s^{(4)}(t)\right]^2 dt = \int_a^b s^{(4)}(t)\cdot(s'''(t))'dt = -\int_a^b s^{(5)}(t)\cdot s'''(t) =$$

$$= -\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} s^{(5)}(t)\cdot s'''(t)dt = -\sum_{k=1}^{n} c_k \int_{t_{k-1}}^{t_k} s'''(t)dt =$$

$$= -\sum_{k=1}^{n} c_k\left[s''(t_k) - s''(t_{k-1})\right] = 0,$$

where $c_k = s^{(5)}(t)\mid_{I_k},\ k = \overline{1,n}$.

It follows that $s^{(4)} = 0$, for all $t \in [a, b]$. Since $s \in \mathcal{P}_3$ on I_0 and on I_{n+1} and $s \in C^4(\mathbf{R})$, it follows that $s^{(4)}(t) = 0$ for all $t \in \mathbf{R}$, implying $s'' \in \mathcal{P}_1$. As $s''(t_k) = 0$, $k = 0, 1,\dots, n$, $(n \ge 3)$ we conclude that $s''(t) = 0$ for all $t \in \mathbf{R}$.

Finally, taking into account the equalities $s(a) = s(b) = 0$, one obtains $s(t) = 0$ for all $t \in \mathbf{R}$, implying that all the coefficients in representation (1) are null. This shows that the homogeneous system associated to (7) has only the trivial solution.

Assertion b) can be proved similarly, supposing that the function s given by (1) verifies conditions (6). □

COROLLARY 3. *There exist the systems of functions*

$$\mathcal{S} = \{s_0, s_1, S_0, S_1,\dots, S_n\} \subset S_5(\Delta_n),$$

$$\mathcal{U} = (u_0, u_1, U_0, U_1,\dots, U_n) \subset S_5(\Delta_n)$$

verifying the conditions

$$s_0(a) = 1,\ s_0(b) = 0,\ s_0''(t_k) = 0,\ k = \overline{0,n},$$

$$s_1(a) = 0,\ s_1(b) = 1,\ s_1''(t_k) = 0,\ k = \overline{0,n},$$

$$S_k(a) = 0,\ S_k(b) = 0,\ k = \overline{0,n};\ S_k''(t_j) = \delta_{kj},\ k, j = \overline{0,n},$$

$$u_0(a) = 1,\ u_0'(a) = 0,\ u_0''(t_k) = 0,\ k = \overline{0,n},$$

$$u_1(a) = 0,\ u_1'(a) = 1,\ u_1''(t_k) = 0,\ k = \overline{0,n},$$

$$U_k(a) = 0,\ U_k'(a) = 0,\ k = \overline{0,n};\ U_k''(t_j) = \delta_{kj},\ k, j = \overline{0,n}.$$

If $f, h: \mathbf{R} \to \mathbf{R}$ verify the conditions of Theorem 2, then the functions s_f and s_h admit the representations

(8) $$s_f(t) = s_0(t)\cdot f(a) + s_1(t)\cdot f(b) + \sum_{k=0}^{n} S_k(t)\cdot f''(t_k),\ t \in \mathbf{R},$$

(9) $$s_h(t) = u_0(t)\cdot h(a) + u_1(t)\cdot h'(a) + \sum_{k=0}^{n} U_k(t)\cdot h''(t_k),\ t \in \mathbf{R}.$$

Remark 1. *By Corollary 3 it follows that the set $S_5(\Delta_n)$ is a (real) linear space of dimension $n + 3$ and $\mathcal{S}$ and $\mathcal{U}$ are two bases in $S_5(\Delta_n)$.*

Some properties of the space $S_5(\Delta_n)$ will be presented in what follows.

Let

(10) $$W_2^4(\Delta_n) := \left\{ \begin{matrix} g:[a,b] \to \mathbf{R},\ g'''\ \text{abs. cont. on}\ I_k,\ k = \overline{1,n} \\ \text{and}\ g^{(4)} \in L_2[a,b] \end{matrix} \right\},$$

(11) $W^4_{2,f}(\Delta_n) := \left\{g \in W^4_2(\Delta_n) : g''(t_k) = f''(t_k),\ k = \overline{0,n}\right\}$,

(12) $W^4_{2,f,D}(\Delta_n) := \left\{g \in W^4_{2,f}(\Delta_n) : g(t_0) = f(t_0),\ g(t_n) = f(t_n)\right\}$,

(13) $W^4_{2,h,C}(\Delta_n) := \left\{g \in W^4_{2,h}(\Delta_n) : g(t_0) = h(t_0),\ g'(t_0) = h'(t_0)\right\}$.

Then we have

THEOREM 4. a) *If* $s \in S_5(\Delta_n) \cap W^4_{2,f,D}(\Delta_n)$, *then*

(14) $$\left\|s^{(4)}\right\|_2 \le \left\|g^{(4)}\right\|_2, \text{ for all } g \in W^4_{2,f,D}(\Delta_n).$$

b) *If* $s \in S_5(\Delta_n) \cap W^4_{2,h,C}(\Delta_n)$, *then*

(15) $$\left\|s^{(4)}\right\|_2 \le \left\|g^{(4)}\right\|_2, \text{ for all } g \in W^4_{2,h,C}(\Delta_n).$$

Proof. We have

$$0 \le \left\|g^{(4)} - s^{(4)}\right\|_2^2 = \int_a^b \left[g^{(4)}(t) - s^{(4)}(t)\right] dt =$$

$$= \int_a^b \left[g^{(4)}(t)\right]^2 dt - \int_a^b \left[s^{(4)}(t)\right]^2 dt - 2\int_a^b s^{(4)}(t)\left[g^{(4)}(t) - s^{(4)}(t)\right] dt.$$

But

$$\int_a^b s^{(4)}(t)\left[g^{(4)}(t) - s^{(4)}(t)\right] dt = s^{(4)}(t)\left[g'''(t) - s'''(t)\right]\Big|_a^b -$$

$$-\int_a^b s^{(5)}(t)\left[g'''(t) - s'''(t)\right] dt = -\int_a^b s^{(5)}(t)\left[g'''(t) - s'''(t)\right] dt =$$

$$= -\sum\nolimits_{k=1}^n \int_{t_{k-1}}^{t_k} s^{(5)}(t)\left[g'''(t) - s'''(t)\right] dt =$$

$$= -\sum\nolimits_{k=1}^n C_k\left[g''(t_k) - s''(t_k) - \left(g''(t_{k-1}) - s''(t_{k-1})\right)\right] = 0,$$

where $C_k = s^{(5)}(t)\Big|_{I_k}$, $k = 1, 2, \ldots, n$.

It follows $\left\|g^{(4)}\right\|_2^2 - \left\|s^{(4)}\right\|_2^2 \ge 0$, which is equivalent to (14).

Inequalities (15) can be proved by a similar argument. □

THEOREM 5. a) *If* $f \in W^4_2(\Delta_n)$ *and* $s_f \in S_5(\Delta_n)$ *verify conditions* (4) *from Theorem* 2, *then*

(16) $$\left\|s_f^{(4)} - f^{(4)}\right\|_2 \le \left\|s^{(4)} - f^{(4)}\right\|_2, \text{ for all } s \in S_5(\Delta_n).$$

b) *If* $h \in W^4_2(\Delta_n)$ *and* $s_h \in S_5(\Delta_n)$ *verify conditions* (6) *from Theorem* 2, *then*

(17) $$\left\|s_h^{(4)} - h^{(4)}\right\|_2 \le \left\|s^{(4)} - h^{(4)}\right\|_2, \text{ for all } s \in S_5(\Delta_n).$$

Proof. In order to prove (16), we use the identity

$$\left\|s^{(4)} - f^{(4)}\right\|_2^2 = \int_a^b \left[s^{(4)}(t) - s_f^{(4)}(t)\right]^2 dt + \int_a^b \left[s_f^{(4)}(t) - f^{(4)}(t)\right]^2 dt +$$

$$+2\int_a^b \left[s^{(4)}(t) - s_f^{(4)}(t)\right] \cdot \left[s_f^{(4)}(t) - f^{(4)}(t)\right] dt$$

and prove that

$$T = \int_a^b \left[s^{(4)}(t) - s_f^{(4)}(t)\right] \cdot \left[s_f^{(4)}(t) - f^{(4)}(t)\right] dt = 0.$$

Indeed, integrating by parts, we find

$$T = \left[s^{(4)}(t) - s_f^{(4)}(t)\right] \cdot \left[s_f'''(t) - f'''(t)\right]\Big|_a^b -$$

$$-\int_a^b \left[s^{(5)}(t) - s_f^{(5)}(t)\right] \cdot \left[s_f'''(t) - f'''(t)\right] dt =$$

$$= -\sum_{k=1}^n c_k(s)\left[s''_k(t_k) - f''(t_k)\right] - \left[s''_f(t_{k-1}) - f''(t_{k-1})\right] = 0,$$

where $c_k(s) = s^{(5)}(t) - s_f^{(5)}(t)$, $t \in I_k$, $k = \overline{1,n}$. (We have used the fact that $(s^{(4)} - s_f^{(4)})(a) = (s^{(4)} - s_f^{(4)})(b) = 0$.)

Therefore,

(18) $$\left\|s^{(4)} - f^{(4)}\right\|_2^2 = \left\|s^{(4)} - s_f^{(4)}\right\|_2^2 + \left\|s_f^{(4)} - f^{(4)}\right\|_2^2,$$

implying that inequality (16) holds.

Similarly, in the identity

$$\left\|s^{(4)} - h^{(4)}\right\|_2^2 = \int_a^b \left[s^{(4)}(t) - s_h^{(4)}(t)\right]^2 dt + \int_a^b \left[s_h^{(4)}(t) - h^{(4)}(t)\right]^2 dt +$$

$$+2\int_a^b \left[s^{(4)}(t) - s_h^{(4)}(t)\right] \cdot \left[s_h^{(4)}(t) - h^{(4)}(t)\right] dt$$

we have (integrating by parts)

$$Q = \int_a^b \left[s^{(4)}(t) - s_h^{(4)}(t) \right] \cdot \left[s_h^{(4)}(t) - h^{(4)}(t) \right] dt = 0,$$

implying that

$$(*) \qquad \left\| s^{(4)} - h^{(4)} \right\|_2^2 = \left\| s^{(4)} - s_h^{(4)} \right\|_2^2 + \left\| s_h^{(4)} - h^{(4)} \right\|_2^2.$$

From this equality it follows (17). □

COROLLARY 6. *If* $f, h \in W_2^4(\Delta_n)$ *and* $s_f, s_h \in S_5(\Delta_n)$ *verify conditions* (4) *and* (6) *from Theorem 2, then*

$$(19) \qquad \left\| f^{(4)} \right\|_2^2 = \left\| s_f^{(4)} \right\|_2^2 + \left\| f^{(4)} - s_f^{(4)} \right\|_2^2,$$

$$(20) \qquad \left\| h^{(4)} \right\|_2^2 = \left\| s_h^{(4)} \right\|_2^2 + \left\| h^{(4)} - s_h^{(4)} \right\|_2^2,$$

$$(21) \qquad \left\| s_f^{(4)} \right\|_2 \le \left\| f^{(4)} \right\|_2,$$

$$(22) \qquad \left\| s_h^{(4)} \right\|_2 \le \left\| h^{(4)} \right\|_2,$$

$$(23) \qquad \left\| f^{(4)} - s_f^{(4)} \right\| \le \left\| f^{(4)} \right\|_2,$$

$$(24) \qquad \left\| h^{(4)} - s_h^{(4)} \right\|_2 \le \left\| h^{(4)} \right\|_2.$$

Proof. Equalities (19) and (20) follow from (18) and (*) for $s \equiv 0$. The remaining inequalities follow from (19) and (20). □

Application. Consider the bilocal linear problem

$$(D) \quad y'' = p(t) \cdot y + q(t), \ t \in [a, b], \quad y(a) = \alpha, \ y(b) = \beta.$$

If p, q are continuous functions on $[a, b]$ and $p(t) > 0$, $t \in [a, b]$, then the problem (*D*) has a unique solution y (see [3], Theorem 10.1, p. 519).

Consider the Cauchy problems

$$(C_1) \quad y'' = p(t)y + q(t), \ t \in [a, b], \quad y(a) = \alpha, \ y'(a) = 0,$$

$$(C_2) \quad y'' = p(t)y, \ t \in [a, b], \quad y(a) = 0, \ y'(a) = 1.$$

The Cauchy problems have unique solutions y_1, y_2, respectively (see [3], Theorem 5.15, p. 263), and the function

$$(25) \qquad y(t) = y_1(t) + \frac{\beta - y_1(b)}{y_2(b)} y_2(t) \ \text{ with } \ y_2(b) \ne 0, \ t \in [a, b],$$

is the solution of the problem (*D*) (see [3]).

Applying Theorem 2b) to the solutions y_1, y_2 of the problems (C_1), (C_2), it follows that there exist the functions $s_{y_1}, s_{y_2} \in S_5(\Delta_n)$ such that

$$(**) \qquad \begin{aligned} s_{y_1}(a) &= \alpha, \ s'_{y_1}(a) = 0, \ s''_{y_1}(t_k) = y''_1(t_k), \ k = \overline{0, n}, \\ s_{y_2}(a) &= 0, \ s'_{y_2}(a) = 1, \ s''_{y_2}(t_k) = y''_2(t_k), \ k = \overline{0, n}. \end{aligned}$$

We call the functions s_{y_1}, s_{y_2} *spline solutions* in $S_5(\Delta_n)$ of the problems (C_1), (C_2), and the function

$$(26) \qquad s_y(t) = s_{y_1}(t) + \frac{\beta - s_{y_1}(b)}{s_{y_2}(b)} s_{y_2}(t), \ s_{y_2}(b) \ne 0, \ t \in [a, b],$$

is called a *spline solution* in $S_5(\Delta_n)$ of the problem (*D*).

THEOREM 7. *Consider the problem*

$$(C) \qquad y'' = p(t)y + q(t), \ t \in [a, b], \quad y(a) = \alpha, \ y'(a) = \gamma,$$

where $p(t) > 0$, $t \in [a, b]$ *and* p, q *are continuous on* $[a, b]$.

If $y \in W_2^4(\Delta_n)$ *is the exact solution of* (*C*) *and* $s_y \in S_5(\Delta_n)$ *is its spline solution (cf. Theorem 2b)), then we have*

$$(27) \qquad \left\| y'' - s''_y \right\|_\infty \le \sqrt{2} \left\| \Delta_n \right\|^{3/2} \cdot \left\| y^{(4)} \right\|_2,$$

where $\left\| \Delta_n \right\| = \max\{ t_k - t_{k-1}, k = \overline{1, n} \}$.

Proof. We have

$$y''(t_i) - s''_y(t_i) = 0, \ i = \overline{0, n}$$

so that, by Rolle's theorem, there exist $t_i^{(1)} \in (t_i, t_{i+1})$, $i = \overline{0, n-1}$ such that

$$y'''\left(t_i^{(1)}\right) - s'''_y\left(t_i^{(1)}\right) = 0, \ i = \overline{0, n-1}.$$

Applying again Rolle's theorem, it follows the existence of $t_i^{(2)} \in \left(t_i^{(1)}, t_{i+1}^{(1)}\right)$, $i = \overline{0, n-2}$ such that

$$y^{(4)}\left(t_i^{(2)}\right) - s_y^{(4)}\left(t_i^{(2)}\right) = 0, \ i = \overline{0, n-2}.$$

The inequalities

$$\left|t_{i+1}^{(1)} - t_i^{(1)}\right| \le 2\|\Delta_n\| \text{ and } \left|t_{i+1}^{(2)} - t_i^{(2)}\right| \le 3\|\Delta_n\|$$

hold for $i = \overline{0, n-2}$ and $i = \overline{0, n-3}$, respectively.

For every $t \in [a, b]$ there is an index $i_0 \in \{0, 1, \ldots, n-1\}$ such that $\left|t - t_{i_0}^{(1)}\right| \le 2\|\Delta_n\|$ so that, taking into account (24), we have

$$\left|y'''(t) - s'''_y(t)\right| = \left|\int_{t_{i_0}^{(1)}}^{t} \left(y^{(4)}(u) - s_y^{(4)}(u)\right)\mathrm{d}u\right| \le$$

$$\le \left|\int_{t_{i_0}^{(1)}}^{t} \mathrm{d}u\right|^{1/2} \cdot \left|\int_{t_{i_0}^{(1)}}^{t} \left[y^{(4)}(u) - s_y^{(4)}\right]^2 \mathrm{d}u\right|^{1/2} \le$$

$$\le \sqrt{2\|\Delta_n\|} \cdot \left|\int_a^b \left[y^{(4)}(u) - s_y^{(4)}(u)\right]^2 \mathrm{d}u\right|^{1/2},$$

$$\le \sqrt{2} \cdot \|\Delta_n\|^{1/2} \cdot \left\|y^{(4)}\right\|_2.$$

Similarly, for every $t \in [a, b]$ there exist $j_0 \in \{0, 1, \ldots, n-1\}$ such that $\left|t - t_{j_0}^{(1)}\right| \le \|\Delta_n\|$, implying

$$\left|y''(t) - s''_y(t)\right| = \left|\int_{t_{g_0}^{(1)}}^{t} \left[y'''(u) - s'''_y(u)\mathrm{d}u\right]\right| \le$$

$$\le \left\|y''' - s'''_y\right\|_\infty \cdot \|\Delta_n\|.$$

It follows that inequality (27) holds. □

COROLLARY 8. *If $y \in W_2^4(\Delta_n)$ is the exact solution of the problem (C), then*

(28) $$\left\|y - s_y\right\|_\infty \le \sqrt{2}(b-a)^2\|\Delta_n\|^{3/2} \cdot \left\|y^{(4)}\right\|_2.$$

Proof. For every $t \in [a, b]$ we have

$$\left|y(t) - s_y(t)\right| = \left|\int_a^t \left(y'(u) - s'_y(u)\right)\mathrm{d}u\right| \le (b-a) \cdot \left\|y' - s'_y\right\|_\infty$$

and

$$\left|y'(t) - s'_y(t)\right| = \left|\int_a^t \left(y''(u) - s''_y(u)\right)\mathrm{d}u\right| \le (b-a) \cdot \left\|y'' - s''_y\right\|_\infty.$$

From these inequalities and from (20) we obtain (28). □

Remark 2. From the proof of Corollary 8 it follows that the inequality

(29) $$\left\|y' - s'_y\right\|_\infty \le \sqrt{2}(b-a)\|\Delta_n\|^{3/2} \cdot \left\|y^{(4)}\right\|_2$$

holds, too.

The approximative determination of the values of the spline solution s_y of the problem (D) on the nodes of the division Δ_n

First observe that the exact solution $y \in W_2^4(\Delta_n)$ of the problem (D) and its spline solution $s_y \in S_5(\Delta_n)$ given by (26) verify

$$\left|y(t) - s_y(t)\right| = \left|y_1(t) + \frac{\beta - y_1(b)}{y_2(b)} y_2(t) - s_{y_1}(t) - \frac{\beta - s_{y_1}(b)}{s_{y_2}(b)} \cdot s_{y_2}(t)\right| \le$$

$$\le \left|y_1(t) - s_{y_1}(t)\right| + \left|\frac{\beta - y_1(b)}{y_2(b)} y_2(t) - \frac{\beta - s_{y_1}(b)}{s_{y_2}(b)} \cdot s_{y_2}(t)\right|$$

for every $t \in [a, b]$, where s_{y_1} and s_{y_2} are determined by the conditions (**).

Using (28), we obtain

$$\frac{\beta - y_1(b)}{y_2(b)} = \frac{\beta - s_{y_1}(b)}{s_{y_2}(b)} + O\left(\|\Delta_n\|^{3/2}\right),$$

showing that

$$\left\|y(t) - s_y(t)\right\| = O\left(\|\Delta_n\|^{3/2}\right).$$

a) *The approximative determination of the solution s_{y_1} on the nodes of the division Δ_n*

Representation (9) yields

$$s_{y_1}(t) = u_0(t) \cdot \alpha + \sum_{k=0}^{n} U_k(t) \cdot y''_1(t_k) =$$

$$= u_0(t) \cdot \alpha + \sum_{k=0}^{n} U_k(t)\left[p(t_k) \cdot y_1(t_k) + q(t_k)\right].$$

Letting

$$v_i := s_{y_1}(t_i), \quad i = \overline{0, n},$$

$$e_i := y_1(t_i) - s_{y_1}(t_i), \quad i = \overline{0, n},$$

one obtains the system

$$s_{y_1}(t_i) = u_0(t_i)\alpha + \sum_{k=0}^{n} U_k(t_i)\left[p(t_k)(e_k + v_k) + q(t_k)\right] =$$

$$= u_0(t_i)\alpha + \sum_{k=0}^{n} U_k(t_i)\left[p(t_k)v_k + q(t_k)\right] + O\left(\|\Delta_n\|^{3/2}\right),$$

$i = \overline{0, n}.$

The approximative values of the spline solution s_{y_1} on the nodes of Δ_n are the solutions v_κ of the linear system

$$v_i = u_0(t_i)\alpha + \sum_{k=0}^{n} U_k(t_i)\left[p(t_k)v_k + q(t_k)\right], \qquad i = \overline{0,n}.$$

b) *The approximative determination of the solution s_{y_2} on the nodes of Δ_n*

Using again representation (9), one obtains

$$s_{y_2}(t) = u_1(t) + \sum_{k=0}^{n} U_k(t)\cdot y''_2(t_k) =$$

$$= u_1(t) + \sum_{k=0}^{n} U_k(t)\cdot p(t_k)\cdot y_2(t_k).$$

Letting

$$w_i := s_{y_2}(t_i), \ i = \overline{0,n},$$

$$e_i := y_2(t_i) - s_{y_2}(t_i), \ i = \overline{0,n},$$

it follows that w_i are the solutions of the system

$$w_i = u_1(t_i) + \sum_{k=0}^{n} U_k(t_i)p(t_k)w_k + O\left(\|\Delta_n\|^{3/2}\right).$$

Therefore, the approximative values of $s_{y_2}(t_i)$ can be obtained from the linear system

(30) $$w_i = u_1(t_i) + \sum_{k=0}^{n} U_k(t_i)p(t_k)\cdot w_k, \ i = \overline{0,n}.$$

The approximative values of the spline solution $s_y \in S_5(\Delta_n)$ on the nodes of the division Δ_n are given by

(31) $$s_y(t_i) = v_i + \frac{\beta - v_n}{w_n} w_i, \ i = \overline{0,n}.$$

A numerical example. The problem

(D) $y'' = 4y, \ t \in [0, 1]$
$y(0) = 1, \ y(1) = e^{-2}$

has the exact solution $y = e^{-2t}$.

The associated Cauchy problems are

(C_1) $y'' = 4y, \ t \in [0, 1]$
$y(0) = 1, \ y'(0) = 0$

(C_2) $y'' = 4y, \ t \in [0, 1]$
$y(0) = 0, \ y'(0) = 1$

and have the exact solutions

$$y_1(t) = \frac{1}{2}\left[e^{2t} + e^{-2t}\right]$$

$$y_2(t) = \frac{1}{4}\left[e^{2t} - e^{-2t}\right]$$

For $n = 5$, let

$\Delta_5 : = \{t_0 = 0, t_1 = 0.2, t_2 = 0.4, t_3 = 0.6, t_4 = 0.8, t_5 = 1\}$.

Using representation (1), one obtains Table 1 for the coefficients of s_{y_1} and s_{y_2}.

Table 1

$n = 5$	s_{y_1}	s_{y_2}
A_0	1	0
A_1	0	1
A_2	2	0
A_3	0.1576268148	0.6678202118
a_0	0.8446081893	0.1257524863
a_1	−0.6853940844	0.07800170745
a_2	−0.6014825811	−0.1770186129
a_3	3.020199906	0.7804651905
a_4	−5.717416684	−1.970643804
a_5	3.139485253	1.163443033

For the values of s_y on the nodes of Δ_5, one uses

$$s_y(t_i) = s_{y_1}(t_i) + \frac{e^{-2} - s_{y_1}(1)}{s_{y_2}(1)} s_{y_2}(t_i), \ i = \overline{0,5}.$$

Table 2 contains the values $s_y(t_i)$, $i = \overline{0,5}$, and the errors

$$E_i = |y(t_i) - s_y(t_i)|, \ i = \overline{0,5}.$$

Table 2

t_i	$s_y(t_i)$	E_i
0	1	0
0.2	0.6708587727	$0.5387267\cdot 10^{-3}$
0.4	0.4506125215	$0.12835574\cdot 10^{-2}$
0.6	0.303315766	$0.21215541\cdot 10^{-2}$
0.8	0.204249434	$0.2352916\cdot 10^{-2}$
1	0.135335284	$0.8\cdot 10^{-9}$

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Received May 15, 1996

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