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ON p-DERIVATIVE-INTERPOLATING SPLINE FUNCTIONS

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Following the ideas from [2] and [3], we define the p-derivative-interpolating spline functions which can be used to approximate the solution of a differential equation of order p ($p \in N, p \geq 1$) with modified argument. For $p = 1$ one obtains the spline functions considered in [2] and [3].

Let

$$\Delta_n : -\infty = t_{-1} < a = t_0 < t_1 < \ldots < t_n = b < t_{n+1} = +\infty$$

be a partition of an interval $[a, b] \subset R$.

DEFINITION 1. *For $n \geq 1, p \geq 1, m \geq 2, m \geq p$ given natural numbers such that $m + p \leq n + 2$, a function $s : R \to R$ satisfying the conditions*

1) $s \in C^{2m+p-2}(R)$,

2) $s\big|_{I_k} \in \mathcal{P}_{2m+p-1}, I_k = [t_{k-1}, t_k), k = 1, 2, \ldots, n,$ *and*

3) $s\big|_{I_0}, s\big|_{I_{n+1}} \in \mathcal{P}_{m+p-1}, I_0 = (t_{-1}, t_0), I_{n+1} = [t_n, t_{n+1})$

is called a spline function of degree $2m + p - 1$. Here $\mathcal{P}_r$ denotes the set of all polynomials of degree at most r.

The set of all spline functions of degree $2m + p - 1$ is denoted by $S_{2m+p-1}(\Delta_n)$.

Remark 1. For $p = 1$ one obtains the set $S_{2m}(\Delta_n)$ of natural polynomial spline functions of even degree $2m$ considered in [2] and [3].

The following representation theorem will imply that the set $S_{2m+p-1}(\Delta_n)$ is an $(n + p + 1)$-dimensional subspace of $C^{2m+p-2}(R)$.

THEOREM 2. *Every element $s \in S_{2m+p-1}(\Delta_n)$ admits the representation*

$$(1) \qquad s(t) = \sum_{i=0}^{m+p-1} A_i t^i + \sum_{k=0}^{n} a_k (t - t_k)_+^{2m+p-1},$$

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where

$$(2)\qquad \sum_{k=0}^{n} a_k t_k^j = 0,\ j = 0,1,\dots,m-1.$$

Proof. Taking into account condition 3) from Definition 1, it follows that $s^{(m+p)}(t) = 0$ for all $t \in I_{n+1}$, giving

$$0 = \sum_{k=0}^{n} (2m+p-1)\dots(m+p-1)a_k(t-t_k)_+^{m-1} = M\sum_{k=0}^{n} a_k(t-t_k)_+^{m-1} =$$

$$= M\sum_{k=0}^{n} a_k\left(\sum_{j=0}^{m-1} C_{m-1}^j t^{m-j-1} t_k^j\right) = M\sum_{j=0}^{m-1}(-1)^j C_{m-1}^j\left(\sum_{k=0}^{n} a_k t_k^j\right)t^{m-j-1},$$

where $M := (2m+p-1)\dots(m+p-1)$.

The above equalities imply that $\sum_{k=0}^{n} a_k t_k^j = 0$, for $j = 0,1,\dots,m-1$. □

THEOREM 3. *Let $q \in N, 0 \le q \le p-1$ and let $f: R \to R$ be a function verifying the conditions*

$$(3)\qquad \begin{cases} f^{(q)}(t_0) = y_0^{(q)}, & q = 0,1,\dots,p-1,\\ f^{(p)}(t_k) = y_k^{(p)}, & k = 0,1,\dots,n. \end{cases}$$

Then there exists a unique spline function $s_f \in S_{2m+p-1}(\Delta_n)$ such that

$$(4)\qquad \begin{cases} s_f^{(q)}(t_0) = y_0^{(q)}, & q = 0,1,\dots,p-1,\\ s_f^{(p)}(t_k) = y_k^{(p)}, & k = 0,1,\dots,n. \end{cases}$$

Proof. Since the spline function s_f admits representation (1), it follows

$$s_f^{(q)}(t) = \sum_{k=0}^{m+p-q-1}\frac{(q+k)!}{k!}A_{q+k}t^k + \sum_{k=0}^{n}\frac{(2m+p-1)!}{(2m+p-q-1)!}a_k(t-t_k)_+^{2m+p-q-1}$$

for $q = 0, 1, \dots, p$.

Imposing conditions (4) to this function, one obtains the system

$$(5)\qquad \begin{cases} \sum_{k=0}^{m+p-q-1}\frac{(q+k)!}{k!}A_{q+k}t_0^k = y_0^{(q)}, & q = \overline{0,p-1}\\ \sum_{k=0}^{m-1}\frac{(p+k)!}{k!}A_{q+k}t_j^k + \sum_{k=0}^{n}\frac{(2m+p-1)!}{(2m-1)!}a_k(t_j-t_k)_+^{2m-1} = y_j^{(p)}, & j = \overline{0,n}\\ \sum_{k=0}^{n} a_k t_k^i = 0, & i = \overline{0,m-1} \end{cases}$$

with $(p+n+1+m)$ equations and $m+p+n+1$ unknowns: $A_0,\dots,A_{m+p-1}$, $a_0,\dots,a_n$.

The system (5) has a unique solution if and only if the associated homogeneous system $\left(y_0^{(q)} = 0, q = 0,1,\dots,p-1; y_j^{(p)} = 0, j = 0,1,\dots,n\right)$ has only the trivial solution.

Denoting by s a function in $S_{2m+p-1}(\Delta_n)$ verifying the homogeneous conditions (4) (i.e., $s^{(q)}(t_0) = 0, q = \overline{0,p-1}; s^{(p)}(t_k) = 0, k = \overline{0,n}$), and integrating by parts, we get

$$\int_{t_0}^{t_n}\left[s^{(m+p)}(t)\right]^2 dt = \sum_{j=0}^{m-2}(-1)^j s^{(m+p+j)}(t)\,s^{(m+p-j-1)}(t)\Big|_{t_0}^{t_n} + (-1)^{m-1}\int_{t_0}^{t_n} s^{(2m+p-1)}(t)s^{(p+1)}(t)dt.$$

Taking into account Definition 1, we further obtain

$$\int_{t_0}^{t_n}\left[s^{(m+p)}(t)\right]^2 dt = (-1)^{m-1}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} s^{(2m+p-1)}(t)s^{(p+1)}(t)dt =$$

$$= (-1)^{m-1}\int_{t_{k-1}}^{t_k} s^{(p+1)}(t)dt = (-1)^{m-1}\sum_{k=1}^{n} c_k\left[s^{(p)}(t_k) - s^{(p)}(t_{k-1})\right] = 0,$$

where $c_k = s^{(2m+p-1)}(t)\big|_{I_k}$, $k = \overline{1,n}$.

It follows that $s^{(m+p)}(t) = 0$ for all $t \in [t_0, t_n]$.

Since $s \in \mathcal{P}_{m+p-1}$ on the intervals I_0 and I_{n+1}, we find that $s^{(m+p)}(t) = 0$, for all $t \in I_0 \cup I_{n+1}$, too, so that by the continuity we get $s^{(m+p)}(t) = 0$, for all $t \in R$.

The equality $s^{(m+p)}(t) = 0$, for all $t \in R$, implies that $s^{(p)} \in \mathcal{P}_{m-1}$ on R. Since $s^{(p)}(t_k) = 0$ for $k = 0,1,\dots,n$, because $m \le n+1$ and $1 \le p \le n-m+1$, it follows $s^{(p)} \equiv 0$ on R. Now, using the conditions $s^{(q)}(t_0) = 0, q = \overline{0,p-1}$, we obtain $s \equiv 0$ on R and, consequently, all the coefficients $A_0,\dots,A_{m+p-1}, a_0,\dots,a_n$ in (1) are null. Taking into account the linear independence of the functions $\left\{1, t, \dots, t^{m+p-1}, (t-t_0)_+^{2m+p-1}, \dots, (t-t_n)_+^{2m+p-1}\right\}$, it follows that the system (5) has a unique solution. □

An immediate consequence of Theorem 3 is the following

COROLLARY 4. *There exists a unique subset of $n+p+1$ spline functions*

$$\{s_0, s_1, \dots, s_{p-1}, S_0, S_1, \dots, S_n\} \subset S_{2m+p-1}(\Delta_n)$$

satisfying the conditions

$$(6)\qquad \begin{cases} s_j^{(q)}(t_0) = \delta_{jq};\ q, j = \overline{0, p-1} \\ s_j^{(p)}(t_k) = 0;\ j = \overline{0, p-1}, k = \overline{1, n}, \end{cases}$$

and, respectively,

$$(7)\qquad \begin{cases} S_k^{(q)}(t_0) = 0;\ q = \overline{0, p-1},\ k = \overline{1, n} \\ S_k^{(p)}(t_i) = \delta_{ki};\ k, i = \overline{0, n}. \end{cases}$$

Let $f \in C^{(p)}(R)$ and $S: C^{(p)}(R) \to S_{2m+p-1}(\Delta_n)$ be the spline operator defined by $S(f) = s_f$.

Obviously, the functions $s_0, s_1, \ldots, s_{p-1}, S_0, S_1, \ldots, S_n$ defined by (6) and (7) form a basis of the space $S_{2m+p-1}(\Delta_n)$. Therefore, s_f has the representation

$$(8)\qquad s_f(t) = \sum_{q=0}^{p-1} s_q(t) f^{(q)}(t_0) + \sum_{k=0}^{n} S_k(t) f^{(p)}(t_k).$$

In order to study the properties of the space $S_{2m+p-1}(\Delta_n)$, we consider the notations

$$(9)\qquad W_2^{m+p}(\Delta_n) := \Big\{ g:[a,b] \to R,\ g^{(m+p-1)} \text{ is abs. cont. on } I_k,\ k = \overline{1,n} \text{ and } g^{(m+p)} \in L_2[a,b] \Big\}$$

$$(10)\qquad W_2^{m+p}([a,b]) := \Big\{ g:[a,b] \to R,\ g^{(m+p-1)} \text{ is abs. cont. on } [a,b] \text{ and } g^{(m+p)} \in L_2[a,b] \Big\}$$

$$(11)\qquad W_{2,f}^{m+p}(\Delta_n) := \Big\{ g \in W_2^{m+p}(\Delta_n),\ g^{(p)}(t_k) = f^{(p)}(t_k),\ k = \overline{0,n} \Big\}$$

$$(12)\qquad W_{2,f^0}^{m+p}(\Delta_n) := \Big\{ g \in W_{2,f}^{m+p}(\Delta_n),\ g^{(q)}(t_0) = f^{(q)}(t_0),\ q = \overline{0, p-1} \Big\}.$$

THEOREM 5. *If $s \in S_{2m+p-1}(\Delta_n) \cap W_{2,f^0}^{m+p}(\Delta_n)$, then the inequality*

$$(13)\qquad \left\| s^{(m+p)} \right\|_2 \le \left\| g^{(m+p)} \right\|_2$$

holds for every $g \in W_{2,f^0}^{m+p}(\Delta_n)$.

Proof. Observe that the last term of the relations

$$0 \le \left\| g^{(m+p)} - s^{(m+p)} \right\|_2^2 = \int_a^b \left[g^{(m+p)}(t) - s^{(m+p)}(t) \right]^2 dt =$$

$$= \int_a^b \left[g^{(m+p)}(t) \right]^2 dt - \int_a^b \left[s^{(m+p)}(t) \right]^2 dt - 2\int_a^b s^{(m+p)}(t) \left[g^{(m+p)}(t) - s^{(m+p)}(t) \right] dt$$

is null. Indeed,

$$T = \int_a^b s^{(m+p)}(t) \left[g^{(m+p)}(t) - s^{(m+p)}(t) \right] dt =$$

$$= \sum_{j=0}^{m-2} (-1)^j s^{(m+p+j)}(t) \left[g^{(m+p-j)}(t) - s^{(m+p-j)}(t) \right] \Big|_{t=a}^{t=b} +$$

$$+ (-1)^{m-1} \int_a^b s^{(2m+p-1)}(t) \left[g^{(p+1)}(t) - s^{(p+1)}(t) \right] dt$$

but $s^{(2m+p-1)}(a) = s^{(m+p-1)}(b) = 0$, $j = \overline{0, m-2}$, and $s^{(2m+p-1)}(t) = c_k$, $k = \overline{1, n}$.

Therefore,

$$T = (-1)^{m-1} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} s^{(2m+p-1)}(t) \left[g^{(p+1)}(t) - s^{(p+1)}(t) \right] dt =$$

$$= (-1)^{m-1} \sum_{k=1}^{n} c_k \left[g^{(p)}(t_k) - s^{(p)}(t_k) - \left(g^{(p)}(t_{k-1}) - s^{(p)}(t_{k-1}) \right) \right] = 0.$$

It follows $0 \ge \left\| s^{(m+p)} \right\|_2 - \left\| g^{(m+p)} \right\|_2$, which is equivalent to (13). □

Remark 2. From the proof of Theorem 5 it is clear that

$$(14)\qquad \left\| f^{(m+p)} \right\|_2^2 = \left\| s_f^{(m+p)} \right\|_2^2 + \left\| f^{(m+p)} - s_f^{(m+p)} \right\|_2^2$$

for $f \in W_2^{m+p}(\Delta_n)$, implying

$$(15)\qquad \left\| s_f^{(m+p)} \right\|_2 \le \left\| f^{(m+p)} \right\|_2$$

$$(16)\qquad \left\| f^{(m+p)} - s_f^{(m+p)} \right\|_2 \le \left\| f^{(m+p)} \right\|_2.$$

THEOREM 6. *Let $f \in W_2^{m+p}(\Delta_n)$ and $s_f \in S_{2m+p-1}(\Delta_n)$ be given by Theorem 3. Then the inequality*

$$(17)\qquad \left\| f^{(m+p)} - s_f^{(m+p)} \right\|_2 \le \left\| f^{(m+p)} - s^{(m+p)} \right\|_2$$

holds for all $s \in S_{2m+p-1}(\Delta_n)$.

Proof. The last term of the identity

$$\left\|s^{(m+p)}-f^{(m+p)}\right\|_2^2=\int_a^b\left[s^{(m+p)}(t)-s_f^{(m+p)}(t)\right]^2\mathrm{d}t+$$

$$+\int_a^b\left[s_f^{(m+p)}-f^{(m+p)}\right]^2\mathrm{d}t+2\int_a^b\left[s^{(m+p)}(t)-s_f^{(m+p)}(t)\right]\left[s_f^{(m+p)}(t)-f^{(m+p)}(t)\right]\mathrm{d}t$$

is null. Indeed,

$$T_1=\int_a^b\left[s^{(m+p)}(t)-s_f^{(m+p)}(t)\right]\left[s_f^{(m+p)}(t)-f^{(m+p)}(t)\right]\mathrm{d}t=$$

$$=\sum_{j=0}^{m-2}(-1)^j\left[s^{(m+p+j)}(t)-s_f^{(m+p+j)}(t)\right]\left[s_f^{(m+p-j)}(t)-f^{(m+p-j)}(t)\right]\Big|_{t=a}^{t=b}+$$

$$+(-1)^{m-1}\int_a^b\left[s^{(2m+p-1)}(t)-s_f^{(2m+p-1)}(t)\right]\left[s_f^{(p+1)}(t)-f^{(p+1)}(t)\right]\mathrm{d}t$$

and, since $\left(s^{(m+p+j)}-s_f^{(m+p+j)}\right)(a)=\left(s^{(m+p+j)}-s_f^{(m+p+j)}\right)(b)=0,\ j=\overline{0,m-2}$ and $s^{(2m+p-1)}(t)-s_f^{(2m+p-1)}(t)=c_k(s)$ (constants) for all $t\in I_k, k=\overline{1,n}$, one obtains

$$T_1=(-1)^{m-1}\sum_{k=0}^{n}c_k(s)\left(\left[s_f^{(p)}(t_k)-f^{(p)}(t_k)\right]-\left[s_f^{(p)}(t_{k-1})-f^{(p)}(t_{k-1})\right]\right)=0.$$

Therefore,

$$\left\|s^{(m+p)}-f^{(m+p)}\right\|_2^2=\left\|s^{(m+p)}(t)-s_f^{(m+p)}(t)\right\|_2^2-\left\|s_f^{(m+p)}(t)-f^{(m+p)}(t)\right\|_2^2,$$

implying (17). □

1. APPLICATIONS

In the following we shall use the spline functions in S_{2m+p-1} to approximate the solutions of Cauchy problems for differential equations with modified argument of order $p(p\geq 1)$.

Consider the Cauchy problem

$$(P)\begin{cases}y^{(p)}(t)=f\big(t,y(t),y(\varphi(t))\big),\ t\in[a,b]\\ y^{(q)}(a)=m_q,\ q=\overline{0,p-1}\\ \varphi:[a,b]\to[a,b]\end{cases}$$

and suppose that the conditions ensuring the existence and uniqueness of the solution y of this problem are fulfilled (see [4]).

From Theorem 3 one obtains

THEOREM 7. *If y is the exact solution of the problem (P), then there exists a unique spline function $s_y\in S_{2m+p-1}(\Delta_n)$ verifying the conditions*

$$(18)\qquad\begin{cases}s_y^{(q)}(t_0)=y^{(q)}(a)=m_q, & q=\overline{0,p-1}\\ s_y^{(p)}(t_k)=y^{(p)}(t_k), & k=\overline{0,n}.\end{cases}$$

Consider the notations

$$(19)\qquad\begin{cases}y(t_k)=y_k;\ y(\varphi(t_k))=\overline{y}_k, & k=\overline{0,n}\\ s_y^{(p)}(t_k)=f(t_k,y_k,\overline{y}_k), & k=\overline{0,n}.\end{cases}$$

Using Corollary 4, we obtain

THEOREM 8. *If $\{s_0,s_1,\ldots,s_{p-1},S_0,S_1,\ldots,S_n\}$, the basis of $S_{2m+p-1}(\Delta_n)$, is given by Corollary 4, then the spline function $s_y\in S_{2m+p-1}(\Delta_n)$ given by Theorem 7 admits the representation*

$$(20)\qquad s_y(t)=\sum_{q=0}^{p-1}s_q(t)m_q+\sum_{k=0}^{n}S_k(t)f(t_k,y_k,\overline{y_k}).$$

We call the spline function s_y given by (20) *the approximate spline solution* of the problem (*P*).

THEOREM 9. *If $y\in W_2^{m+p}[a,b]$ is the exact solution of the problem (P) and s_y is its approximate spline solution, then*

$$(21)\qquad\left\|y^{(m+p-r)}-s_y^{(m+p-r)}\right\|_\infty\leq\sqrt{m}(m-1)\ldots(m-r+1)\|\Delta_n\|^{r-\frac{1}{2}}\left\|y^{(m+p)}\right\|_2$$

$$r=2,3,\ldots,m;\ \|\Delta_n\|=\max\{t_i-t_{i-1},i=\overline{1,n}\}.$$

Proof. Since $y^{(p)}(t_i)-s_y^{(p)}(t_i)=0, i=\overline{0,n}$, by an application of Rôlle's theorem we obtain the existence of points $t_i^{(1)}\in(t_i,t_{i+1}), i=\overline{0,n-1}$, such that

$$y^{(p+1)}\left(t_i^{(1)}\right)-s_y^{(p+1)}\left(t_i^{(1)}\right)=0,\ i=\overline{0,n-1}.$$

Applying again Rôlle's theorem, it follows that there exists $t_i^{(m-1)}\in\left(t_i^{(m-2)},t_{i+1}^{(m-2)}\right)$, $i=\overline{0,n-m+1}$, such that

$$y^{(m+p+1)}\left(t_i^{(m-1)}\right) - s_y^{(m+p+1)}\left(t_i^{(m-1)}\right) = 0,\ i = \overline{0, n-m+1}.$$

It is obvious that

$$\left|t_{i+1}^{(k)} - t_i^{(k)}\right| \le k\|\Delta_n\|,\ k = \overline{0, m-1}.$$

It follows that for every $t \in [a, b]$ there exists i_0 such that $\left|t - t_{i_0}^{(m-1)}\right| \le m\|\Delta_n\|$ and, consequently,

$$\left|y^{(m+p-1)}(t) - s_y^{(m+p-1)}(t)\right| \le \left|\int_{t_{i_0}^{(m-1)}}^{t} \left(y^{(m+p)}(u) - s_y^{(m+p)}(u)\right) du\right| \le$$

$$\le \left|\int_{t_{i_0}^{(m-1)}}^{t} du\right|^{\frac{1}{2}} \left|\int_{t_{i_0}^{(m-1)}}^{t} \left(y^{(m+p)}(u) - s_y^{(m+p)}(u)\right)^2 du\right|^{\frac{1}{2}} \le$$

$$\le \sqrt{m\|\Delta_n\|} \left|\int_{t_{i_0}^{(m-1)}}^{t} \left(y^{(m+p)}(u) - s_y^{(m+p)}(u)\right)^2 du\right|^{\frac{1}{2}} \le$$

$$\le \sqrt{m\|\Delta_n\|} \left|\int_a^b \left(y^{(m+p)}(u) - s_y^{(m+p)}(u)\right)^2 du\right|^{\frac{1}{2}} \le \sqrt{m}\|\Delta_n\|^{\frac{1}{2}} \left\|y^{(m+p)}\right\|_2 .$$

The last inequality from above follows from (16).

Therefore,

$$\left\|y^{(m+p-1)} - s_y^{(m+p-1)}\right\|_\infty \le \sqrt{m}\|\Delta_n\|^{\frac{1}{2}} \left\|y^{(m+p)}\right\|_2 .$$

Similarly, for every $t \in [a, b]$ we can find an index i_0 such that $\left|t - t_{i_0}^{(m-2)}\right| \le (m-1)\|\Delta_n\|$ so that

$$\left|y^{(m+p-2)}(t) - s_y^{(m+p-2)}(t)\right| \le \left|\int_{t_{i_0}^{(m-2)}}^{t} \left(y^{(m+p-1)}(u) - s_y^{(m+p-1)}(u)\right) du\right| \le$$

$$\le \left\|y^{(m+p-1)} - s_y^{(m+p-1)}\right\|_\infty \left|t - t_{i_0}^{(m-2)}\right| \le \sqrt{m}(m-1)\|\Delta_n\|^{1+\frac{1}{2}} \left\|y^{(m+p)}\right\|_2$$

for all $t \in [a, b]$, implying

$$\left\|y^{(m+p-2)} - s_y^{(m+p-2)}\right\|_\infty \le \sqrt{m}(m-1)\|\Delta_n\|^{1+\frac{1}{2}} \left\|y^{(m+p)}\right\|_2$$

We obtain, in general,

$$\left\|y^{(m+p-r)} - s_y^{(m+p-r)}\right\|_\infty \le \sqrt{m}(m-1)\ldots(m-r+1)\|\Delta_n\|^{r-\frac{1}{2}} \left\|y^{(m+p)}\right\|_2$$

$r = 2, 3, \ldots, m$. □

Remark 3. For $r = m$ one obtains the evaluation

(22) $$\left\|y^{(p)} - s_y^{(p)}\right\|_\infty \le \sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}} \left\|y^{(m+p)}\right\|_2 .$$

COROLLARY 10. *The inequality*

(23) $$\left\|y - s_y\right\|_\infty \le (b-a)^p \sqrt{m}(m-1)!\|\Delta_n\|^{m-\frac{1}{2}} \left\|y^{(m+p)}\right\|_2$$

holds for every $y \in W_2^{m+p}[a, b]$.

Proof. We have

$$\left|y(t) - s_y(t)\right| = \left|\int_{t_0}^{t} \left(y'(u) - s'_y(u)\right) du\right| \le \left|t - t_0\right| \left\|y' - s'_y\right\|_\infty \le (b-a)\left\|y' - s'_y\right\|_\infty .$$

Similarly,

$$\left\|y' - s'_y\right\|_\infty \le (b-a)^2 \left\|y'' - s''_y\right\|_\infty$$

and, finally,

$$\left\|y - s_y\right\|_\infty \le (b-a)^p \left\|y^{(p)} - s_y^{(p)}\right\|_\infty .$$

Now (23) follows from (22). □

COROLLARY 11. *The relation*

(24) $$\lim_{\|\Delta_n\| \to 0} \left\|y^{(k)} - s_y^{(k)}\right\|_\infty = 0,\ k = p, p+1, \ldots, p+m-2$$

holds for every $y \in W_2^{m+p}[a, b]$.

Proof. It follows immediately from (21).

Now we shall show how the above results can be applied to obtain the approximate spline solution (20) of the problem (*P*).

Denote

$$(25)\quad \begin{cases} w_i := s_y(t_i) := \sum_{q=0}^{p-1} s_q(t_i) y^{(q)}(t_0) + \sum_{k=0}^{n} S_k(t_i) f(t_k, y_k, \overline{y}_k), \ i = \overline{0,n} \\ \overline{w}_i := s_y(\varphi(t_i)) := \sum_{q=0}^{p-1} s_q(\varphi(t_i)) y^{(q)}(t_0) + \sum_{k=0}^{n} S_k(\varphi(t_i)) f(t_k, y_k, \overline{y}_k), \ i = \overline{0,n} \end{cases}$$

and let

$$e_i := e(t_i) = y(t_i) - s_y(t_i), \ i = 0,1,\ldots,n$$

$$\overline{e}_i := e(\varphi(t_i)) = y(\varphi(t_i)) - s_y(\varphi(t_i)), \ i = 0,1,\ldots,n$$

denote the deviation of the approximate spline solution s_y from the exact solution y of the problem (P), on the knots t_i and $\varphi(t_i)$, $i = 0, 1, \ldots, n$.

We have

$$y_k = w_k + e_k$$

$$\overline{y}_k = \overline{w}_k + \overline{e}_k$$

for $k = 0, 1, \ldots, n$ and the system (25) can be written in the following form

$$\begin{cases} w_i = \sum_{q=0}^{p-1} s_q(t_i) y^{(q)}(t_0) + \sum_{k=0}^{n} S_k(t_i) f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k), \ i = 0,1,\ldots,n \\ \overline{w}_i = \sum_{q=0}^{p-1} s_q(\varphi(t_i)) y^{(q)}(t_0) + \sum_{k=0}^{n} S_k(\varphi(t_i)) f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k), \ i = 0,1,\ldots,n \end{cases}$$

with w_i and $\overline{w}_i$, $i = 0, 1, \ldots, n$ as unknowns.

If the derivatives of the function $f(t,u,v), f: D \subset R^3 \to R; (D \subset [a,b] \times R^2)$ with respect to u, v are continuous, then

$$f(t_k, w_k + e_k, \overline{w}_k + \overline{e}_k) = f(t_k, w_k, \overline{w}_k) + \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial y_k} e_k + \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial \overline{y}_k} \overline{e}_k,$$

where

$$\min(w_k, w_k + e_k) \le \xi_k \le \max(w_k, w_k + e_k)$$

$$\min(\overline{w}_k, \overline{w}_k + \overline{e}_k) \le \eta_k \le \max(\overline{w}_k, \overline{w}_k + \overline{e}_k).$$

One obtains the system

$$(26)\quad \begin{cases} w_i = \sum_{q=0}^{p-1} s_q(t_i) m_q + \sum_{k=0}^{n} S_k(t_i) f(t_k, w_k, \overline{w}_k) + E_i \\ \overline{w}_i = \sum_{q=0}^{p-1} s_q(\varphi(t_i)) m_q + \sum_{k=0}^{n} S_k(\varphi(t_i)) f(t_k, w_k, \overline{w}_k) + \overline{E}_i, \end{cases}$$

where

$$E_i = \sum_{k=0}^{n} S_k(t_i) \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial y_k} e_k + \sum_{k=0}^{n} S_k(t_i) \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial \overline{y}_k} \overline{e}_k, \ i = \overline{0,n}$$

$$\overline{E}_i = \sum_{k=0}^{n} S_k(\varphi(t_i)) \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial y_k} e_k + \sum_{k=0}^{n} S_k(\varphi(t_i)) \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial \overline{y}_k} \overline{e}_k, \ i = \overline{0,n}.$$

Supposing that the derivatives of $f(t, u, v)$ with respect to u, v are bounded on D, there exist $M_1, N_1 > 0$ such that

$$\left| \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial y_k} \right| \le M_1, \quad \left| \frac{\partial f(t_k, \xi_k, \eta_k)}{\partial \overline{y}_k} \right| \le N_1, \ k = \overline{0,n}$$

and, taking into account Remark 3, we deduce that

$$E_i = O\left(\|\Delta_n\|^{m-\frac{1}{2}} \right), \overline{E}_i = O\left(\|\Delta_n\|^{m-\frac{1}{2}} \right),$$

consequently $E_i \to 0$, $\overline{E}_i \to 0$, for $\|\Delta_n\| \to 0$.

Now, neglecting the quantities E_i, $\overline{E}_i$, $i = \overline{0,n}$, we obtain the following system of $2n+2$ equations

$$(27)\quad \begin{cases} w_i = \sum_{q=0}^{p-1} s_q(t_i) m_q + \sum_{k=0}^{n} S_k(t_i) f(t_k, w_k, \overline{w}_k), \ i = \overline{0,n} \\ \overline{w}_i = \sum_{q=0}^{p-1} s_q(\varphi(t_i)) m_q + \sum_{k=0}^{n} S_k(\varphi(t_i)) f(t_k, w_k, \overline{w}_k), \ i = \overline{0,n} \end{cases}$$

with $2n + 2$ unknowns: $w_0, w_1, \ldots, w_n, \overline{w}_0, \overline{w}_1, \ldots, \overline{w}_n$.

Consider the notations

$$[s] = \begin{bmatrix} s_0(t_0) & \dots & s_{p-1}(t_0) & & & \\ \dots & \dots & \dots & & 0 & \\ s_0(t_n) & \dots & s_{p-1}(t_n) & & & \\ & & & s_0(\varphi(t_0)) & \dots & s_{p-1}(\varphi(t_0)) \\ & 0 & & \dots & \dots & \dots \\ & & & s_0(\varphi(t_n)) & \dots & s_{p-1}(\varphi(t_n)) \end{bmatrix}$$

$$m = (m_0, m_1, \dots, m_{p-1}, m_0, m_1, \dots, m_{p-1})^T$$

$$[S] = \begin{bmatrix} S_0(t_0) & \dots & S_n(t_0) & & & \\ \dots & \dots & \dots & & 0 & \\ S_0(t_n) & \dots & S_n(t_n) & & & \\ & & & S_0(\varphi(t_0)) & \dots & S_n(\varphi(t_0)) \\ & 0 & & \dots & \dots & \\ & & & S_0(\varphi(t_n)) & \dots & S_n(\varphi(t_n)) \end{bmatrix}$$

$$W = (w_0, w_1, \dots, w_n, \overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)^T$$

$$f(W) = \left(f(t_0, w_0, \overline{w}_0), \dots, f(t_n, w_n, \overline{w}_n), f(t_0, w_0, \overline{w}_0), \dots, f(t_n, w_n, \overline{w}_n)\right)^T.$$

Using these notations, the system (27) can be written in the following form

(28) $$W = [s]m + [S]f(W).$$

We obtain

(29) $$W = H(W),$$

where $H : \Omega \to \Omega$, $H(W) = [s]m + [S]f(W)$ and the domain $\Omega \subset R^{2n+2}$ will be specified below:

Let $M = \max\{|m_q| : q = 0, 1, \dots, p-1\}$ and $N > 0$ such that $|f(t_k, w_k, \overline{w}_k)| \le N$, $k = \overline{0, n}$. The set $\Omega \subset R^{2n+2}$ is defined by

(30) $$\Omega = \Big\{W \in R^{2n+2}; W = (w_0, w_1, \dots, w_n, \overline{w}_0, \overline{w}_1, \dots, \overline{w}_n)^T,$$

$$|w_i| \le M \sum_{q=0}^{p-1} |s_q(t_i)| + N \sum_{k=0}^{n} |S_k(t_i)|,\ i = \overline{0, n}$$

$$|\overline{w}_i| \le M \sum_{q=0}^{p-1} |s_q(\varphi(t_i))| + N \sum_{k=0}^{n} |S_k(\varphi(t_i))|,\ i = \overline{0, n}\Big\}.$$

Since H is a continuous application of the compact convex Ω into itself, it follows that equation (29) has at least one solution $W^* \in \Omega$.

Let $[F]$ be the following diagonal matrix

$$[F] = \begin{bmatrix} \frac{\partial f(t_0, w_0, \overline{w}_0)}{\partial w_0} & & & & & \\ & \dots & & & 0 & \\ & & \frac{\partial f(t_n, w_n, \overline{w}_n)}{\partial w_n} & & & \\ & & & \frac{\partial f(t_0, w_0, \overline{w}_0)}{\partial \overline{w}_0} & & \\ & 0 & & & \dots & \\ & & & & & \frac{\partial f(t_n, w_n, \overline{w}_n)}{\partial \overline{w}_n} \end{bmatrix}$$

Then

$$\frac{dH}{dW} = [S][F].$$

THEOREM 12. *If* $\left\|\frac{dH}{dW}\right\| \le Q < 1$, *then the solution* $W^* \in \Omega$ *of equation* (29) *is unique and can be found by the iterative process*

$$W^{(k)} := H(W^{(k-1)}),\ k = 1, 2, 3, \dots,$$

where $W^{(0)} \in \Omega$ *is arbitrarily chosen. Furthermore, the following evaluation holds*

$$\left\|W^* - W^{(k)}\right\| \le \frac{Q^k}{1-Q}\left\|W^{(1)} - W^{(0)}\right\|.$$

Proof. It is an immediate consequence of the Banach's fixed point theorem. □

Remark 4. We have $\left\|\frac{dH}{dW}\right\| \le \|[S]\| \|[F]\| < 1$ if and only if $\|[F]\| < \frac{1}{\|[S]\|}$.

If the norm of a matrix $[A] = (a_{ij})_{i,j=\overline{1,2n+2}}$ is defined by

$$\|A\| = \left(\sum_{i=1}^{2n+2} \sum_{j=1}^{2n+2} |a_{ij}|^2 \right)^{\frac{1}{2}},$$

then

$$\|[F]\| = \left(\sum_{i=0}^{n} \left(\frac{\partial f(t_i, w_i, \overline{w}_i)}{\partial w_i} \right)^2 + \sum_{i=0}^{n} \left(\frac{\partial f(t_i, w_i, \overline{w}_i)}{\partial w_i} \right)^2 \right)^{\frac{1}{2}} = \sqrt{2n+2}M_0,$$

where $M_0 = \max\{M_1, N_1\}$. It follows that $\|[F]\| < \frac{1}{\|[S]\|}$ if and only if $M_0 < \frac{1}{\sqrt{2n+2}\|[S]\|}$.

2. A NUMERICAL EXAMPLE

Consider the Cauchy problem

$$(P)\begin{cases} y'''(t) = \frac{1}{2} e^{t-\frac{t}{2}} y\left(\frac{t}{2}\right) + \frac{1}{2} y(t), \ t \in [0,1] \\ y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1. \end{cases}$$

Its exact solution is

$$y(t) = e^t.$$

Table 1 contains the calculated values of the spline approximating function on indicated points as well as the absolute errors (in the case $m = 3, p = 3, n = 4$).

Table 1

x	$s_y(x)$	$\lvert e(x)\rvert$
0	1	0
1/10	1.105170755	$0.136 \cdot 10^{-6}$
2/10	1.221400733	$0.2025 \cdot 10^{-5}$
3/10	1.349851256	$0.7552 \cdot 10^{-5}$
4/10	1.491808464	$0.16234 \cdot 10^{-4}$
5/10	1.648697829	$0.23442 \cdot 10^{-4}$
6/10	1.822098038	$0.20762 \cdot 10^{-4}$
7/10	2.013753559	$0.852 \cdot 10^{-6}$
8/10	2.225585132	$0.44204 \cdot 10^{-4}$
9/10	2.459697653	$0.94542 \cdot 10^{-4}$
1	2.718385876	$0.104048 \cdot 10^{-3}$

REFERENCES

1. H. Akça and Gh. Micula, *Numerical solution of differential equations of* nth *order with deviating argument by spline functions*, Bul. St. Univ. Baia Mare, Seria B, Matematică-Informatică VII, *1-2* (1991), 47-54.
2. P. Blaga, R. Gorenflo and Gh. Micula, *Even degree spline technique for numerical solution of delay differential equations*, Freie Universität Berlin, Preprint No. A-15 (1996), Serie A-Mathematik.
3. Gh. Micula, P. Blaga and M. Micula, *On even degree polynomial spline functions with applications to numerical solution of differential equations with retarded argument*, Technische Hochschule Darmstadt, Preprint No. 1771 (1995), Fachbereich Mathematik.
4. A. I. Rus, *Principles and Applications of Fixed Point Theory*, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).
5. Ma Tsoy-Wo, *Classical Analysis on Normed Spaces*, World Scientific, Singapore-New Jersey-London-Hong Kong, 1994.

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